PHYSICS 378 General Relativity and Cosmology (2005)

Assignment 1 Solutions

1. Assume that one of the coordinate transformation equations for a reference frame S and a second reference frame S' moving at a velocity v_x with respect to each the first is of the form

$$x' = Ax^2 + Bt,$$

i.e. non-linear. Suppose that in the frame of reference S' a rod is lying at rest along the X' axis with one end at $x'_1 = 0$ the other at $x'_2 = l'$, i.e. the rod has a length l'. If the positions of the ends of the rod are measured at a time t in S, determine from this transformation law the coordinates x_1 and x_2 of the ends of the rod at this time t. Hence show that the length $x_2 - x_1$ of the rod as measured in S depends on when this length is measured.

SOLUTION

If the positions of the two ends of the rods are measured at the same time t in S, then, for the end of the rod at $x'_1 = 0$ in S', the position as measured in S will be x_1 such that

$$x'_1 = 0 = Ax_1^2 + Bt$$

while the position of the other end, at x'_2 in S' will, in S, be at x_2 where

$$x'_2 = Ax_2^2 + Bt.$$

The length of the rod as measured in S will then be

$$x_2 - x_1 = \left(\sqrt{-Bt} - \sqrt{x'_2 - Bt}\right)/A$$

which is clearly a result that depends on the time at which the measurement is performed, a result that is inconsistent with the homogeneity assumptions about time (and space).

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2. Write down the Lorentz transformation equations that give the coordinates (x', t') in S' of an event that has coordinates (x, t) in S where S' is moving with a velocity v_x with respect to S. By treating these equations as a pair of simultaneous equations, invert the transformation by solving for x' and t' and interpret your result. Hence propose a quicker way of inverting the transformation.

SOLUTION

The Lorentz transformation equations are

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(t - vx/c^2) \\ \text{where} \quad \gamma &= \frac{1}{\sqrt{1-(v/c)^2}}. \end{aligned}$$

These equations can be solved for x and t by use of Cramer's rule. First write in matrix form:

$$\begin{pmatrix} \gamma & -\gamma v \\ -\gamma v/c^2 & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x' \\ t' \end{pmatrix}.$$

The solutions are then

$$x = \frac{\begin{vmatrix} x' & -\gamma v \\ t' & \gamma \end{vmatrix}}{\begin{vmatrix} \gamma & -\gamma v \\ -\gamma v/c^2 & \gamma \end{vmatrix}} = \frac{\gamma x' + \gamma v t'}{\gamma^2 - \gamma^2 (v/c)^2} = \gamma(x' + vt')$$

and

$$t = \frac{\begin{vmatrix} \gamma & x' \\ -\gamma v/c^2 & t' \end{vmatrix}}{\begin{vmatrix} \gamma & -\gamma v \\ -\gamma v/c^2 & \gamma \end{vmatrix}} = \frac{\gamma t' + \gamma v/c^2 x'}{\gamma^2 - \gamma^2 (v/c)^2} = \gamma(t' + vx'/c^2).$$

A more direct way of obtaining this result can be seen by noting that if S' is moving with a velocity v_x with respect to S, then S will be moving with a velocity $-v_x$ with respect to S. Thus the transformation from S' to S can be obtained by the transformation from S to S' by changing the sign of v_x and swapping $x \leftrightarrow x'$, $t \leftrightarrow t'$.

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3. A reference frame S' passes a second reference frame S with a velocity of $0.6c$ in the X direction. Clocks are adjusted in the two frames so that when $t = t' = 0$ the origins of the two reference frames coincide.

   (a) An event occurs in S with space-time coordinates $x_1 = 50\text{m}$, $t_1 = 2.0 \times 10^{-7}\text{s}$. What are the coordinates of this event in S'?

   (b) A second event occurs at $x_2 = 10\text{m}$, $t_2 = 3.0 \times 10^{-7}$ s. What are the coordinates of this event in S'?

   (c) What is the time interval between the events as measured in S and S'? Is this difference an example solely of time dilation? Give reasons for your conclusion.

SOLUTION

(a) The Lorentz transformation is defined by the set of equations

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(t - vx/c^2) \\ \text{where} \quad \gamma &= \frac{1}{\sqrt{1-(v/c)^2}}. \end{aligned}$$

Consequently, if the coordinates of an event in S is $x_1 = 50\text{m}$, $t_1 = 2.0 \times 10^{-7}\text{s}$, and the relative velocities of the two frames is $v = 0.6c$, then

$$\begin{aligned} x_1' &= \frac{1}{\sqrt{1-(0.6)^2}}(50 - 0.6 \times 3 \times 10^8 \times 2.0 \times 10^{-7}) = 17.5 \text{ m} \\ t_1' &= \frac{1}{\sqrt{1-(0.6)^2}}(2.0 \times 10^{-7} - 0.6 \times 50/3 \times 10^8) = 1.25 \times 10^{-7} \text{ s}. \end{aligned}$$

(b) Using the Lorentz transformation once again for the event $x_2 = 10\text{m}$, $t_2 = 3.0 \times 10^{-7}\text{s}$ in S gives

$$\begin{aligned} x_2' &= \frac{1}{\sqrt{1-(0.6)^2}}(10 - 0.6 \times 3 \times 10^8 \times 3.0 \times 10^{-7}) = -55 \text{ m} \\ t_2' &= \frac{1}{\sqrt{1-(0.6)^2}}(3.0 \times 10^{-7} - 0.6 \times 10/3 \times 10^8) = 3.5 \times 10^{-7} \text{ s} \end{aligned}$$

(c) The time interval between the two events, as measured in S', is $\Delta t' = t_2' - t_1' = 2.25 \times 10^{-7}$ s and $\Delta t = t_2 - t_1 = 10^{-7}$ s as measured in S. The difference is a result, in part, of time dilation, but there is also a contribution due to the fact that the clocks synchronized within one reference frame are not synchronized when observed from another reference frame. This can be seen from the Lorenz transformation equation

$$\Delta t' = \gamma \Delta t - v\Delta x/c^2.$$

If the positions of the two events were at the same spatial point in S, i.e. $\Delta x = 0$, then the time difference Δt would be the time difference as registered by a *single* clock positioned at this single point in S, and then this last transformation would reduce to $\Delta t' = \gamma \Delta t$, the time dilation formula. However, Δt is the time difference of the events as measured by two clocks in S that are separated by a distance Δx, and because of this separation, these clocks are measured to be out of synchronization by an amount $-\gamma v \Delta x/c^2$ by the clocks in S'. Adding this out of synchronization amount to the time dilation contribution then gives the final time difference measured in S'.

4. A person at the origin of an inertial reference frame S observes a rod of proper length l_0 moving towards him at a speed v. He notes that the rod takes a time T to pass him.

   (a) As measured in S', the rest frame of the rod, how long will it take for the observer situated at the origin of S take to pass along the length of the rod? [Hint: answering this question does not require a knowledge of special relativity.]

   (b) Assuming that when the front end of the rod passes the observer, the clocks in S and in the rest frame S' of the rod are both set to read zero, fill in the missing information in the following space-time coordinates (x, t) and $x', t')$ of the two specified events:

      (i) Event E_1, the front end of the rod passing the observer at the origin of S, occurs at $(0, ?)$ in S and $(0, 0)$ in S'.

      (ii) Event E_2, the tail end of the rod passing the observer at the origin of S, occurs at $(0, ?)$ in S and $(l_0, ?)$ in S'.

   (c) Using the Lorentz transformation, establish a relation between the coordinates of event E_2.

   (d) Hence prove that the velocity of the rod is given by

$$v = \frac{l_0/T}{\sqrt{1 + \left[(l_0/T)/c\right]^2}}$$

SOLUTION

   (a) The observer situated at the origin of the reference frame S will be measured in S' to be approaching at a speed of v. Since the rod has length l_0 as measured in S', this observer at the origin of S will take a time l_0/v to pass along its length.

   (b) (i) Since the clocks on both S and S' are synchronized to read zero when the front end of the rod passes the origin of S, then event E_1 will be observed to occur at $(0, 0)$ in S.

      (ii) The tail end of the rod will pass the observer at the origin of S at time T, hence event E_2 will occur at the spacetime point $(0, T)$ as measured in S. The rod must be assumed to be moving in the negative x direction with respect to S, i.e. the frame of reference S' (the rest frame of this rod) will be moving with a velocity $-v$ with respect to S. With this direction of relative motion, the tail of the rod will be positioned at l_0, and, from part (a), will be observed to pass the position of the observer in S at time l_0/v. Hence, in S', event E_2 will occur at the spacetime point $(l_0, l_0/v)$.

   (c) Event E_2 has coordinates $(0, T)$ as measured in S, and $(l_0, l_0/v)$ as measured in S'. Substituting the coordinates of event E_2 into either of the the Lorentz transformation equations

$$t' = \gamma(t + v/c^2 x) \quad \text{or} \quad x' = \gamma(x + vt).$$

   gives $l_0/v = \gamma T$. Using $\gamma = 1/\sqrt{1 - v^2/c^2}$ and solving for v gives the required result, that is

$$v = \frac{l_0/T}{\sqrt{1 + \left[(l_0/T)/c\right]^2}}.$$

5. A fluorescent tube, stationary in a reference frame S, is arranged so as to light up simultaneously (in S) along its entire length l_0 at the time t. By considering as two simultaneous events in S the lighting up of two parts of the tube an infinitesimal distance Δx apart, determine the temporal and spatial separation of these two events in another frame of reference S' moving with a velocity v parallel to the orientation of the tube. Hence describe what is observed from this other frame of reference.

SOLUTION

Suppose the tube is lying along the x axis, at rest in the reference frame S, with one end at $x = 0$, and the other at $x = l_0$. Further suppose the origin of the second reference frame, moving with a speed v with respect to the first, coincides with the origin of the first at a time $t = t' = 0$.

We can identify two events: the lighting up of the tube at x at time t and at $x + \Delta x$ also at time t in S. The coordinates of these two events in S' will then be x' at time t' and $x' + \Delta x'$ at time $t' + \Delta t'$.

These two sets of coordinates can be related by the Lorentz transformation:

$$\begin{aligned} x' &= \gamma(x - vt) & t' &= \gamma(t - vx/c^2) \\ x' + \Delta x' &= \gamma\big(x + \Delta x - vt\big) & t' + \Delta t' &= \gamma\big(t + v(x + \Delta x)/c^2)\big) \end{aligned}$$

Subtracting the two sets of equations then gives

$$\Delta x' = \gamma(\Delta x - v\Delta t) \qquad \Delta t' = -\gamma v \Delta x/c^2.$$

Although some conclusions can be drawn from each of these equations, it is best to combine them to find the *speed* with which the lighted up region spreads along the tube, at least as far as it is measured in S'. This speed is given by

$$u_{\text{measured}} = \frac{\Delta x'}{\Delta t'} = \frac{\Delta x}{-v\Delta x/c^2} = -\frac{c^2}{v}$$

which is a velocity that will always be greater than the speed of light! There is no conflict with relativity as the events that give rise to this apparent superluminal velocity were *prearranged* to occur simultaneously in S, i.e. the simultaneous lighting up of the tube was not due to the passage of some kind of physical signal along the tube.

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6. Two Star Wars killer satellites are stationary in the S frame at points on the X-axis separated by a distance d. They fire laser pulses at one another simultaneously. From the point of view of the frame of reference of an observer space shuttle moving with a velocity u relative to S, show that one satellite fires a time $\gamma ud/c^2$ before the other.

SOLUTION

Suppose the Star Wars satellites are positioned at x_1 and x_2 as measured in S, and that they both fire their lasers at the instant t. These two events will then have the spacetime coordinates (x_1', t_1') and (x_2', t_2') as measured in S', the frame of reference of the space shuttle.

The times at which these events occur according to S are then given in terms of their coordinates in S' are then

$$t'_n = \gamma(t - ux_n/c^2); \quad n = 1, 2.$$

The time difference between the events as measured in S is then

$$t_2 - t_1 = -\gamma u(x_2 - x_1)/c^2 = -\gamma ud/c^2.$$

In other words, one satellite is observed to fire a time $\gamma ud/c^2$ before the other. The order depends on the sign of u, and on the sign of the difference $x_2 - x_1$.

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7. An observer on earth sees two UFOs travelling directly towards each other with a velocity $0.7c$ relative to the observer on earth. According to an observer in one of the UFOs, how fast is the other UFO approaching? How fast is the distance between the UFOs diminishing according to the observer on earth?

SOLUTION

Suppose one UFO is travelling with a velocity $u_x = -0.7c$ as measured in the frame of reference S of the observer. We require the speed of this UFO as measured in the frame of reference of the other UFO which is at rest in a reference frame S'. This reference frame is moving with a velocity $v_x = 0.7c$ with respect to S. What we are then after is u'_x, given by

$$u'_x = \frac{u_x - v_x}{1 - u_x v_x/c^2} = \frac{0.7c - (-0.7c)}{1 - (-0.7c)(0.7c)/c^2} = 0.9396c.$$

As far as the observer on earth is concerned, what will be observed is two UFOs moving towards each other at a speed of 0.7c each. So in a time Δt, one UFO will move a distance $0.7c\Delta t$ to the right, the other a distance $-0.7c\Delta t$ to the left. The distance between them will then have diminished by an amount $1.4c\Delta t$. Consequently, the distance between the two UFOs will be diminishing at a rate of 1.4c.
