Questions from past exam papers.

1. (a) (8 marks)

   The Hamiltonian of an electron in magnetic field is: $\hat{H} = -\omega \hat{S}_z$ where $\hat{S}_z$ is the operator of the z-component of spin. An electron has been prepared in the state $|S_x^+\rangle$ at $t = 0$, where $|S_x^+\rangle$ is the eigenstate of spin x-component with the eigenvalue of $\frac{1}{2}\hbar$.

   (i) Evaluate the state of this electron at $t = T$.

   (ii) Explain why this electron is said to be in a non-stationary state.

   (iii) Describe the physical experiment that needs to be done in order to bring this electron to a stationary state.

   (b) (6 marks)

   Without a detailed calculation explain what is meant by spin precession.

   (c) (6 marks)

   For an electron in a magnetic field prove that the expectation value of the z-component of spin in an energy eigenstate does not depend on time.

2. (a) (6 marks)

   (i) For the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ check that $[\hat{H}, \hat{p}] = i\hbar\, dV/dx$.

   (ii) Interpret this result by developing the analogy between classical and quantum mechanics.

   (b) (6 marks)

   A quantum system can exist in two states $|0\rangle$ and $|1\rangle$, which are normalised eigenstates of an observable A with eigenvalues 0 and 1 respectively. ($\hat{A}$ has only these two eigenvalues). The quantum system is prepared in the state $|0\rangle$ at $t = 0$ and is allowed to freely evolve as dictated by the time evolution operator with a time-independent Hamiltonian $\hat{H}$, but without any other external interference. Then the observable A is measured at $t = T$, but its value is lost. The system is further allowed to freely evolve as before, and later the observable A is measured for the second time, at $t = 2T$. Assuming that the probability of observing the value 0 in the first measurement is known and equals x^2, where $0 < x^2 < 1$ calculate the probability that the second measurement gives the value of 0. Give a detailed explanation of your reasoning.

   (c) (8 marks)

   (i) Explain the difference between the quantum state and each of its following representations:

   - the position representation (or the wavefunction in position representation);
   - the momentum representation (or the wavefunction in position representation);
   - the energy representation.

   You may wish to use analogies drawn from the finite-dimensional vector spaces.

(ii) What is the relationship between the wavefunction $\Psi(x)$ in position representation and the wavefunction $\Psi(p)$ in momentum representation, of the same state $|\Psi\rangle$.

(iii) What is the physical interpretation of the quantities $|\Psi(x)|^2 dx$ and $|\Psi(p)|^2 dp$ (for one-dimensional problems).

3. (a) (8 marks)

The vector potential of a single mode (linearly polarized, single frequency) of a perfect cavity of length L and cross-sectional area A can be shown to be given by $\mathbf{A}(z,t) = A_0(t)\mathbf{i}\sin kz$, where $\mathbf{i}$ is a unit vector in the x direction, k is an integer multiple of π/L, and with a similar expression for the electric field.

(i) From Maxwell's equations, show that

$$\frac{dA_0(t)}{dt} = -E_0(t) \quad \text{and} \quad \frac{dE_0(t)}{dt} = c^2k^2A_0(t).$$

(ii) The classical Hamiltonian of the field inside the cavity can be shown to be

$$H = \tfrac{1}{2}[P^2 + \omega^2 Q^2]$$

where

$$P = -\sqrt{\tfrac{1}{2}\epsilon_0 LA}\,E_0 \quad \text{and} \quad Q = \sqrt{\tfrac{1}{2}\epsilon_0 LA}\,A_0$$

Show that Hamilton's equations of motion obtained from this Hamiltonian are identical to the equations obtained in part (i). Hence describe how a quantum theory of this single mode field can be constructed.

(b) (8 marks)

The electric field operator in a cavity of length L and cross-sectional area A is given by

$$\hat{\mathbf{E}}(z) = -i\sqrt{\frac{\hbar\omega}{\epsilon_0 LA}}(\hat{a}^\dagger - \hat{a})\sin kz\,\mathbf{i}.$$

Calculate the expectation value of the field strength, and the uncertainty in the field strength, if the cavity field is in the vacuum state. Provide a physical interpretation of your results.

(c) (4 marks)

In the Hamiltonian describing the interaction of a two level atom, excited state $|e\rangle$, ground state $|g\rangle$ with this single mode field, there appears a term

$$i\hbar\Omega(z)(\hat{a} + \hat{a}^\dagger)|e\rangle\langle g| - i\hbar\Omega^*(z)(\hat{a} + \hat{a}^\dagger)|g\rangle\langle e|.$$

Describe the physical processes that are represented here, and indicate those terms that would be excluded under the rotating wave approximation, stating why they could be neglected.

4. A quantum system can exist in two states $|0\rangle$ and $|1\rangle$, which are normalised eigenstates of an observable $\hat{A}$ with eigenvalues 0 and 1 respectively. The Hamiltonian operator is defined by

$$\hat{H}|0\rangle = \alpha|0\rangle + \beta|1\rangle$$
$$\hat{H}|1\rangle = \beta|0\rangle + \alpha|1\rangle$$

where α, β are real.

(a) (6 marks)

Determine the time evolution operator for this system and its matrix in the $|0\rangle, |1\rangle$ basis.

(b) (6 marks)

If the system is in the state $|1\rangle$ at time $t = 0$, write down the state of this system at time $t = T$.

(c) (8 marks)

The same system is made to evolve from the state $|0\rangle$ at the time t=0 and is further subjected to a sequence of measurements. First, the observable $\hat{A}$ is measured at $t = T$, but the value is lost. The system is then allowed to further evolve for an additional period of T and then the system energy is measured. Calculate the expectation value of this energy measurement. What is the relation between this expectation value and the average value of energy obtained in energy measurements of an ensemble of systems, identically prepared as this one?

5. (a) (6 marks)

A spin-$\frac{1}{2}$ system is subjected to a uniform magnetic field aligned along the z-axis. Show that the z-component of spin is the conserved in the motion. Is the x-component of spin conserved as well?

(b) (8 marks)

Describe the procedure of canonical quantisation. Explain how the quantum mechanical description of a system with a classical analogue converges to their classical description, when we start to consider larger systems.

(c) (6 marks)

(i) Define the uncertainty of an observable in terms of the dispersion of the results of repeated measurements of an observable.

(ii) What is the link between the commutation relation between two observables and the uncertainty principle?

(iii) How does the uncertainty principle manifest itself in experiments on quantum systems?

6. (a) (4 marks)

What is the relationship between the wavefunctions of a particle in a confining potential in the position and momentum representation?

(b) (4 marks)

Write down the state of a free particle with the momentum $\hbar k_1$ in one dimension. Using the translation operator translate this state by a. What can you say about the translated state?

(c) (12 marks)

Show that the eigenstates in a quantum harmonic oscillator have either even or odd parity.

7. (a) (8 marks)

Using the information provided at the front of the paper, describe how the quantization of the electromagnetic field inside a single mode cavity is carried out.

(b) (6 marks)

(i) Prove the commutation rule $[\hat{a}, \hat{a}^\dagger] = 1$.

(ii) Show that the quantum Hamiltonian can be written in the form $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.

(iii) How are the eigenstates $|n\rangle$ to be interpreted?

(c) (6 marks)

The electric field operator in a cavity of length L and cross-sectional area A is given by

$$\hat{\mathbf{E}}(z) = -i\sqrt{\frac{\hbar\omega}{\epsilon_0 LA}}(\hat{a}^\dagger - \hat{a})\sin kz\,\mathbf{i}.$$

Calculate the expectation value of the field strength, and the uncertainty in the field strength, if the cavity field is in the vacuum state. Provide a physical interpretation of your results.

8. The Hamiltonian of a one dimensional harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}.$$

Assume that the oscillator has been prepared in a superposition of the ground and first excited states, $|\phi\rangle = a|0\rangle + b|1\rangle$, at $t = 0$.

(a) (6 marks) Find the state of the oscillator at $t = T$ and evaluate its average kinetic energy

$$E_K(T) = \left\langle \frac{\hat{p}^2}{2m} \right\rangle$$

in this state.

(b) (6 marks) Explain why the oscillator is said to be in a non-stationary state, and describe how one can bring the oscillator to a stationary state (energy eigenstate).

(c) (6 marks) For the state at time $t = T$, what is the probability of finding the oscillator in the ground state?

(d) (7 marks) If we perform a measurement on the position of the oscillator at time $t = T$, what is the probability of finding the oscillator in the position eigenstate $|x_0\rangle$. After this measurement, what is the state of the oscillator?

9. (a) (3 marks)

The ket vectors $\{|\varphi_n\rangle; n = 1, 2, \ldots\}$ form a complete, orthonormal set of basis states. Explain what the terms “complete”, “orthonormal”, and “basis” mean.

(b) (4 marks)

One of the following ket vectors is not a physically permissible state. Which one is acceptable, and explain why.

$$|\phi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}|\varphi_n\rangle; \quad |\psi\rangle = \sum_{n=0}^{\infty} \frac{1}{n+1}|\varphi_n\rangle$$

(c) (7 marks)

State whether each of the following statements is true or false, and if false, either write down the correct statement or explain why the statement is false.

(i) For any operator $\hat{A}$, if $\hat{A}|\psi\rangle = |\phi\rangle$, then $\langle\psi|\hat{A} = \langle\phi|$.

(ii) The ket vectors $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ represent different physical states of a system.

(iii) If $\hat{A}$ is a linear operator, then $\hat{A}(|\psi\rangle + |\phi\rangle) = \hat{A}|\psi\rangle + \hat{A}|\phi\rangle$.

(iv) If $\hat{A} = \hat{A}^\dagger$ then $\hat{A}$ has real eigenvalues.

(v) If $\hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A} = \hat{1}$ then $\hat{A}$ is Hermitean.

(vi) The state $|\psi\rangle \doteq \begin{pmatrix} \sqrt{2} \\ i \end{pmatrix}$ is normalized to unity.

(vii) $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$

(d) (6 marks)

If $\hat{A}$ is a Hermitean operator with eigenstates $\{|a_n\rangle; n = 1, 2, \ldots\}$ and associated eigenvalues $a_n, n = 1, 2, \ldots$, show that

$$\langle\psi|f(\hat{A})|\psi\rangle = \sum_n f(a_n)|\langle\psi|a_n\rangle|^2$$

where $f(x)$ is a function that can be expanded as a power series in x.

10. (a) (3 marks)

Write down the definition of the time evolution operator $\hat{U}(t_1, t_2)$ and show that it is unitary.

(b) For evolution over an infinitesimal time interval $(t, t + \delta t)$, the time evolution operator is given by $\hat{U}(t + \delta t, t) = \hat{1} - i\hat{H}\delta t/\hbar$. From this expression:

(i) (2 marks)

Show that $\hat{H}$ is Hermitean.

(ii) (2 marks)

Derive the Schrödinger equation $\hat{H}|\psi(t)\rangle = i\hbar\dfrac{d}{dt}|\psi(t)\rangle$.

(c) The Hamiltonian of a certain spin half system is of the form

$$\hat{H} \doteq \begin{pmatrix} \langle+|\hat{H}|+\rangle & \langle+|\hat{H}|-\rangle \\ \langle-|\hat{H}|+\rangle & \langle-|\hat{H}|-\rangle \end{pmatrix} = \begin{pmatrix} -\epsilon & -V \\ -V & \epsilon \end{pmatrix}$$

where $|\pm\rangle$ are the eigenstates of $\hat{S}_z$ with eigenvalues $\pm\frac{1}{2}\hbar$.

(i) (8 marks)

If at time t the state of the system is $|\psi(t)\rangle = c_+(t)|+\rangle + c_-(t)|-\rangle$, show that $c_+(t)$ satisfies the equation

$$\ddot{c}_+ + \hbar^{-2}(\epsilon^2 + V^2)c_+ = 0.$$

(ii) (5 marks)

The system is observed at $t = 0$ to be in the eigenstate of $\hat{S}_z$ with eigenvalue $+\frac{1}{2}\hbar$. What is the probability of observing it to be in this state at some other time t?

11. (a) (8 marks)

State, without proof, what the mathematical and physical significance is of two observables $\hat{A}$ and $\hat{B}$ (i) commuting, (ii) not commuting. Your answer should include comments concerning the link between the commutation relation between two observables and the uncertainty principle.

(b) (8 marks)

The space displacement operator $\hat{\mathcal{D}}(a)$ is defined such that $\hat{\mathcal{D}}(a)|x\rangle = |x+a\rangle$, and can be shown to be given by $\hat{\mathcal{D}}(a) = e^{i\hat{p}a/\hbar}$.

(i) Show that $[\hat{x}, \hat{p}] = i\hbar$.

(ii) Show that $\hat{p}$ is a Hermitean operator.

(iii) What observable can $\hat{p}$ be identified with?

(c) (4 marks)

Describe briefly the procedure of canonical quantisation for a system consisting of a single particle moving in one dimension under the action of a potential.

12. (a) (12 marks)

State whether each of the following statements is true or false, and if false, either write down the correct statement or explain why the statement is false. [In the following, the ket vectors $\{|\varphi_n\rangle; n = 1, 2, \ldots\}$ form a complete, orthonormal set of basis states for a certain system $\mathcal{S}$.]

(i) $\langle\varphi_n|\varphi_m\rangle = \frac{1}{2}(\delta_{n,m+1} + 2\delta_{nm} + \delta_{n,m-1})$.

(ii) The system $\mathcal{S}$ can be prepared in a state $|\psi\rangle$ for which $\langle\varphi_n|\psi\rangle = 0$ for every basis state $|\varphi_n\rangle$.

(iii) The state $|\psi\rangle$ of $\mathcal{S}$ defined by

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}|\varphi_n\rangle$$

is a physically permissible state.

(iv) The probability of finding the system $\mathcal{S}$ in the state $|\varphi_1\rangle$ when prepared in the state $|\psi\rangle = \frac{1}{3}|\varphi_1\rangle + \frac{2}{3}|\varphi_2\rangle$ is $\frac{1}{3}$.

(v) For any operator $\hat{A}$, if $\hat{A}|\psi\rangle = |\phi\rangle$, then $\langle\psi|\hat{A} = \langle\phi|$.

(vi) The operator $\hat{A}$ defined such that $\hat{A}(a|\psi\rangle + b|\phi\rangle) = b|\psi\rangle + a|\phi\rangle$ for all states $|\psi\rangle$ and $|\phi\rangle$ and all complex numbers a and b is not a linear operator.

(vii) If $\hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A} = \hat{1}$ then $\hat{A}$ is Hermitean.

(b) (8 marks)

A certain operator $\hat{x}$ is Hermitean with a continuous eigenvalue spectrum, $-\infty < x < +\infty$.

(i) The orthogonality condition for the eigenstates of $\hat{x}$ is $\langle x|x'\rangle = \delta(x-x')$. What is the quantity $\delta(x-x')$ known as and state two of its important properties.

(ii) What is the completeness relation for the eigenstates of $\hat{x}$?

(iii) Show that the inner product of two state vectors $|\psi\rangle$ and $|\phi\rangle$ can be written in the form

$$\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty} \psi(x)^* \phi(x) dx.$$

(iv) For a system in the state $|\psi\rangle$ such that $\langle x|\psi\rangle = A\exp(-x^2 - ikx)$ where k is real, what is the probability of obtaining a result in the range x to $x + dx$?

$$\left[\text{Note that } \int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\pi/a}.\right]$$

13. (a) (8 marks)

State, without proof, what the mathematical and physical significance is of two observables $\hat{A}$ and $\hat{B}$ (i) commuting, (ii) not commuting. Your answer should include comments concerning the link between the commutation relation between two observables and the uncertainty principle.

(b) (12 marks)

A certain molecular system can exist in two states $|-\rangle$ and $|+\rangle$ which are normalized eigenstates of the Hamiltonian $\hat{H}$ for the system with eigenvalues $-\frac{1}{2}\hbar\omega$ and $\frac{1}{2}\hbar\omega$ respectively. The atomic system also possesses a dipole moment represented by the operator $\hat{D} = i\mu[|+\rangle\langle-| - |-\rangle\langle+|]$ where μ is a real number.

(i) What are the eigenvalues and eigenvectors of $\hat{D}$?

(ii) Show that $[\hat{H}, \hat{D}] \neq 0$.

Measurements are made in quick succession of the energy, then the dipole moment, and then the energy of the molecule. The first energy measurement gives the result $-\frac{1}{2}\hbar\omega$. What is the probability of getting the same result on the second measurement of energy? Briefly comment on your result in light of (ii).

14. Consider a simple harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}),$$

and energy eigenstates $|n\rangle, n = 0, 1, 2, \ldots$ satisfying

$$\hat{H}|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle\,.$$

Recall that $\hat{a}, \hat{a}^\dagger$ are defined as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})\,, \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p})\,.$$

(a) (5 marks)

Solve the Schrödinger equation for an energy eigenstate

$$i\hbar\frac{\partial}{\partial t}|n(t)\rangle = \hat{H}|n(t)\rangle\,,$$

to give $|n(t)\rangle$ as an explicit function of time. Explain why energy eigenstates are sometimes called “stationary states”.

(b) (10 marks)

If the system is initially (time $t = 0$) in a coherent state $|\alpha\rangle$ for some complex number α, defined in the energy eigenstate basis by

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,,$$

what is the state of the system at time $t > 0$, in terms of

(i) energy eigenstates?

(ii) coherent states?

(c) (5 marks)

In any coherent state $|\alpha\rangle$, calculate the uncertainty Δx. You may use the useful eigenvalue relations

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \,, \quad \langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^* \,.$$

15. (a) (12 marks)

State whether each of the following statements is true or false, and if false, either write down the correct statement or explain why the statement is false. [In the following, the ket vectors $\{|\varphi_n\rangle; n = 1, 2, \ldots\}$ form a complete, orthonormal set of basis states for a certain system $\mathcal{S}$.]

(i) $\langle\varphi_n|\varphi_m\rangle = \frac{1}{2}(\delta_{n,m+1} + 2\delta_{nm} + \delta_{n,m-1})$.

(ii) The system $\mathcal{S}$ cannot be prepared in a state $|\psi\rangle$ for which $\langle\varphi_n|\psi\rangle = 0$ for every basis state $|\varphi_n\rangle$.

(iii) The probability of finding the system $\mathcal{S}$ in the state $|\varphi_1\rangle$ when prepared in the state $|\psi\rangle = \frac{1}{3}|\varphi_1\rangle + \frac{2}{3}|\varphi_2\rangle$ is $\frac{1}{3}$.

(iv) For any operator $\hat{A}$, if $\hat{A}|\psi\rangle = |\phi\rangle$, then $\langle\psi|\hat{A}^\dagger = \langle\phi|$.

(v) The state $|\psi\rangle \doteq \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$ has zero norm.

(vi) The operator $\hat{A}$ defined such that $\hat{A}|\psi\rangle = \dfrac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}}$ for all $|\psi\rangle$ is a linear operator.

(vii) If $\hat{A}$ is unitary then $\langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle = \langle\psi|\psi\rangle$ for all states $|\psi\rangle$.

(b) (8 marks)

A certain operator $\hat{x}$ is Hermitean with a continuous eigenvalue spectrum, $-\infty < x < +\infty$.

(i) The orthogonality condition for the eigenstates of $\hat{x}$ is $\langle x|x'\rangle = \delta(x - x')$. What is the quantity $\delta(x - x')$ known as and state two of its important properties.

(ii) What is the completeness relation for the eigenstates of $\hat{x}$?

(iii) Write an expression for the operator $\hat{x}$ in ket-bra notation, and check your answer by evaluating $\hat{x}|x\rangle$.

(iv) For a system in the state $|\psi\rangle$ such that $\langle x|\psi\rangle = A\exp(-x^2 - ikx)$ where k is real, what is the probability of obtaining a result in the range x to $x + dx$?

$$\left[\text{Note that } \int_{-\infty}^{+\infty} e^{-ax^2}dx = \sqrt{\pi/a}.\right]$$

16. (a) (6 marks)

State, without proof, what the physical significance is of two observables $\hat{A}$ and $\hat{B}$ (i) commuting, (ii) not commuting, if an alternating sequence of measurements of $\hat{A}$ and $\hat{B}$ is performed.

(b) (6 marks)

The uncertainty relation for two observables $\hat{A}$ and $\hat{B}$ takes the form

$$\Delta\hat{A}\Delta\hat{B} \geq \tfrac{1}{2}|\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle|$$

where $(\Delta\hat{A})^2 = \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2$ and similarly for $\Delta\hat{B}$.

(i) What do $\Delta\hat{A}$ and $\Delta\hat{B}$ represent?

(ii) What does the uncertainty relation mean physically. Illustrate your answer in the case of the position and momentum observables for which $[\hat{x}, \hat{p}] = i\hbar$.

(c) (8 marks)

The matrices representing the three components of spin of a spin half particle are, in the $\hat{S}_z$-representation

$$\hat{S}_x = \tfrac{1}{2}\hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \hat{S}_y = \tfrac{1}{2}\hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \hat{S}_z = \tfrac{1}{2}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

(i) Evaluate the commutator $[\hat{S}_x, \hat{S}_y]$. What is the physical meaning of this result?

(ii) Calculate the uncertainties in the x and y components of spin if the system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right]$$

where $|\pm\rangle$ are the usual eigenstates of $\hat{S}_z$ with eigenvalues $\pm\frac{1}{2}\hbar$.

(iii) Many identical copies of this spin system are prepared in the state $|\psi\rangle$ defined above and a measurement of S_x made on each copy. What does the uncertainty in S_x obtained in (ii) imply about the results of the measurement of S_x made on each copy? Answer this question if instead S_y were measured.

(iv) Confirm that the uncertainty principle is satisfied.

17. (a) (3 marks)

Write down the definition of the time evolution operator $\hat{U}(t_1, t_2)$ and show that it is unitary.

(b) (4 marks)

For evolution over an infinitesimal time interval $(t, t + \delta t)$, the time evolution operator is given by $\hat{U}(t + \delta t, t) = \hat{1} - i\hat{H}\delta t/\hbar$. From this expression:

(i) Show that $\hat{H}$ is Hermitean.

(ii) Derive the Schrödinger equation

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle.$$

(iii) Show that if $|\psi(0)\rangle = |E\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue E, then

$$|\psi(t)\rangle = e^{-iEt/\hbar}|\psi(0)\rangle.$$

(c) (13 marks)
A certain molecular system can exist in two states $|-\rangle$ and $|+\rangle$ which are normalized eigenstates of the Hamiltonian $\hat{H}$ for the system with eigenvalues $-\frac{1}{2}\hbar\omega$ and $\frac{1}{2}\hbar\omega$ respectively. The atomic system also possesses a dipole moment represented by the operator $\hat{D} = \mu[|+\rangle\langle-| + |-\rangle\langle+|]$ where μ is a real number.

(i) Show that the eigenvalues of $\hat{D}$ are $\pm\mu$, and that the corresponding eigenvectors are $[|+\rangle \pm |-\rangle]/\sqrt{2}$.

(ii) The molecule is prepared at $t = 0$ in the eigenstate of $\hat{D}$ with eigenvalue μ. What is the expectation value of $\hat{D}$ as a function of time?

18. Consider the Hamiltonian

$$\hat{H} = \hbar g(\hat{a}^\dagger\hat{a})^3\,,$$

where $\hat{a}, \hat{a}^\dagger$ are defined as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})\,, \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p})\,.$$

(a) (5 marks)
Show that the harmonic oscillator energy eigenstates (number states) $|n\rangle$ are also eigenstates of this Hamiltonian. Calculate the eigenvalues.

(b) (10 marks)
If the system is initially (time $t = 0$) in a coherent state $|\alpha\rangle$ for some complex number α, defined in the energy eigenstate basis by

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,,$$

what is the state of the system at time $T = \pi/g$, in terms of

(i) energy eigenstates?

(ii) coherent states?

Hint: If n is an odd number, then n^3 is also an odd number. If n is an even number, then so is n^3.

(c) (5 marks)
Consider the "Schrödinger cat" state

$$|\text{cat}\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |-\alpha\rangle)\,,$$

for α large purely imaginary. If the momentum was measured on many copies of this state, give a qualitative description of the resulting distribution of measurement outcomes.

19. (a) (12 marks)

State whether each of the following statements is true or false, and if false, either write down the correct statement or explain why the statement is false. [In the following, where they occur, the ket vectors $\{|\varphi_n\rangle; n = 1, 2, \dots\}$ form a complete, orthonormal set of basis states for a certain system $\mathcal{S}$.]

(i) $\langle\varphi_n|\varphi_m\rangle = \frac{1}{2}(\delta_{n,m+1} + 2\delta_{nm} + \delta_{n,m-1})$.

(ii) The system $\mathcal{S}$ can be prepared in a state $|\psi\rangle$ for which $\langle\varphi_n|\psi\rangle = 0$ for every basis state $|\varphi_n\rangle$.

(iii) The state $|\psi\rangle$ of $\mathcal{S}$ defined by

$$|\psi\rangle = \sum_{n=0}^{\infty} \sqrt{n+1}|\varphi_n\rangle$$

is a physically permissible state.

(iv) For any operator $\hat{A}$, if $\hat{A}|\psi\rangle = |\phi\rangle$, then $\langle\psi|\hat{A}^\dagger = \langle\phi|$.

(v) The state $|\psi\rangle \doteq \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$ has zero norm.

(vi) The ket vectors $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ represent different physical states of a system.

(vii) The operator $\hat{A}$ defined such that $\hat{A}|\psi\rangle = \dfrac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}}$ for all $|\psi\rangle$ is a linear operator.

(viii) If $\hat{A}$ is Hermitean then $\langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle = \langle\psi|\psi\rangle$ for all states $|\psi\rangle$.

(b) (4 marks)

A system $\mathcal{S}$ is prepared in a state $|a\rangle$, an eigenstate of an observable $\hat{A}$ with eigenvalue a. Expressed in terms of the eigenstates $|b_1\rangle$ and $|b_2\rangle$, with associated eigenvalues b_1 and b_2 respectively, of a second observable $\hat{B}$, this state is given by

$$|a\rangle = 0.6|b_1\rangle + 0.8i|b_2\rangle.$$

Further, it is known that the action of the observable $\hat{B}$ on the state $|a\rangle$ is

$$\hat{B}|a\rangle = 4|b_1\rangle + 2i|b_2\rangle.$$

Suppose a measurement of $\hat{B}$ is made on the system $\mathcal{S}$ when it is in state $|a\rangle$.

(i) What state (or states) could the system end up in after the measurement of $\hat{B}$ is performed?

(ii) What is the probability of the measurement producing the results b_1 or b_2?

(c) (4 marks)

The position operator $\hat{x}$ for a single particle is a Hermitean operator with a continuous eigenvalue spectrum, $-\infty < x < +\infty$.

(i) The orthogonality condition for the eigenstates of $\hat{x}$ is $\langle x|x'\rangle = \delta(x - x')$. What is the quantity $\delta(x - x')$ known as and state two of its important properties.

(ii) What is the completeness relation for the eigenstates of $\hat{x}$?

(iii) Show that $\langle\psi|\hat{x}|\psi\rangle$ can be written in the form

$$\langle\psi|\hat{x}|\psi\rangle = \int_{-\infty}^{+\infty} x|\psi(x)|^2 dx.$$

(iv) For a system in the state $|\psi\rangle$ such that $\langle x|\psi\rangle = A\exp(-x^2 - ikx)$ where k is real, what is the probability of obtaining a result in the range x to $x + dx$?

$$\left[\text{Note that } \int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\pi/a}.\right]$$

20. (a) (6 marks)
State, without proof, what the physical significance is of two observables $\hat{A}$ and $\hat{B}$ (i) commuting, (ii) not commuting, if an alternating sequence of measurements of $\hat{A}$ and $\hat{B}$ is performed.

(b) (6 marks)
The uncertainty relation for two observables $\hat{A}$ and $\hat{B}$ takes the form

$$\Delta\hat{A}\Delta\hat{B} \geq \tfrac{1}{2}|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|$$

where $(\Delta\hat{A})^2 = \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2$ and similarly for $\Delta\hat{B}$.

(i) What do $\Delta\hat{A}$ and $\Delta\hat{B}$ represent?

(ii) What does the uncertainty relation mean physically. Illustrate your answer in the case of the position and momentum observables for which $[\hat{x},\hat{p}] = i\hbar$.

(c) (8 marks) In carbon monoxide CO^- ion, the electron can be found on the carbon atom at $x = 0$ or the oxygen atom at $x = a$, these values being the eigenvalues of the position observable $\hat{x}$ for the electron, with corresponding position eigenstates $|0\rangle$ and $|a\rangle$. The Hamiltonian $\hat{H}$ of the ion is such that

$$\hat{H}|0\rangle = 8|0\rangle + 3|a\rangle, \quad \hat{H}|a\rangle = 3|0\rangle.$$

(i) Is it possible to have exact knowledge at the same time of the position of the electron *and* of its energy? Give reasons for your answer.

(ii) The position of the electron is measured and it is found to be on the carbon atom. The energy of the ion is then measured, and immediately afterwards, the position of the electron remeasured. What is the probability that the electron is found on the carbon atom on this second measurement? Is this result consistent with your conclusion in part (i)?

[Note that $\hat{H}$ has eigenstates

$$|E_+\rangle = \frac{1}{\sqrt{10}}(3|0\rangle + |a\rangle) \quad |E_-\rangle = \frac{1}{\sqrt{10}}(|0\rangle - 3|a\rangle)$$

with eigenvalues $E_+ = 9$ and $E_- = -1$. You do not need to prove these results.]

21. (a) (8 marks)

The time evolution operator for a certain system is given by *unitary* operator $\hat{U}(t_1, t_2) = e^{-i\hat{H}(t_2-t_1)/\hbar}$.

(i) Explain what is meant by the time evolution operator.

(ii) Show that the unitarity of the time evolution operator means that $\hat{H}$ is Hermitean. What observable of the system can $\hat{H}$ be identified with?

(iii) Derive the Schrödinger equation

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle.$$

(iv) Show that if $|\psi(0)\rangle = |E\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue E, then

$$|\psi(t)\rangle = e^{-iEt/\hbar}|\psi(0)\rangle.$$

(b) (4 marks)

A certain molecular system can exist in two states $|-\rangle$ and $|+\rangle$ which are normalized eigenstates of the Hamiltonian $\hat{H}$ for the system with eigenvalues $-\frac{1}{2}\hbar\omega$ and $\frac{1}{2}\hbar\omega$ respectively. The atomic system also possesses a dipole moment represented by the operator $\hat{D} = \mu[|\mu\rangle\langle\mu| - |-\mu\rangle\langle-\mu|]$ where μ is a real number and where $|\pm\mu\rangle = [|+\rangle \pm |-\rangle]/\sqrt{2}$.

The molecule is prepared at $t = 0$ in the eigenstate of $\hat{D}$ with eigenvalue μ. What is the expectation value of $\hat{D}$ as a function of time?

(c) (6 marks)

The space displacement operator $\hat{\mathcal{D}}(a)$ is defined such that $\hat{\mathcal{D}}(a)|x\rangle = |x+a\rangle$, and can be shown to be given by $\hat{\mathcal{D}}(a) = e^{i\hat{p}a/\hbar}$.

(i) What observable can $\hat{p}$ be identified with?

(ii) Show that $[\hat{x}, \hat{\mathcal{D}}(a)] = a\hat{\mathcal{D}}(a)$. [Hint: evaluate the commutator by allowing it to act on the position basis state $|x\rangle$.] Hence show that $[\hat{x}, \hat{p}] = i\hbar$.

(d) (2 marks)

Describe briefly the procedure of canonical quantisation for a classical simple harmonic oscillator for which the classical potential energy is $\frac{1}{2}kx^2$.