MACQUARIE UNIVERSITY

Department of Physics
Division of ICS

**Summary of Some Important Properties of Bras and Kets**

Given the following concepts:

1. A ket (or ket vector or state vector, or sometimes just state), written $|\ldots\rangle$, is a description of the state of a system, summarized in the '...' as a list of all the information known about the system at one instant in time without mutual interference or contradiction, and is often used to describe the initial or given state of the system in the sense of item 3a below.

2. A bra (or bra vector or state vector, or sometimes just state), written $\langle\ldots|$, is also a description of the state of a system, as in item 1, and is often used to specify the final *observed* state of the system in the sense of item 3a below.

3. (a) The quantity written $\langle\phi|\psi\rangle$ is the probability amplitude of the system being observed in the state $\langle\phi|$ given that it was in the state $|\psi\rangle$;

   (b) $|\langle\phi|\psi\rangle|^2$ is the probability of the system being observed in the state $\langle\phi|$ given that it was in the state $|\psi\rangle$.

4. A set of ket vectors $\{|n\rangle; n = 1, 2, \ldots\}$ consisting of *all* the mutually exclusive, distinguishable states the system could be found in as determined by exhaustive measurement of the values of some physical property[1] of a system, e.g. as would occur when measuring the possible values for the $z$-component of spin, or the position of an electron in an $O_2^-$ ion, or the positions of the slits that a particle could pass through in a multi-slit interference experiment. This set can also be understood as representing *all* the possible mutually exclusive, distinguishable, intermediate states the system could pass through as it passed from some initial state to an observed final state. [Note: $|n\rangle$ is a generic symbol for the state – $n$ would be replaced by an appropriate label for a specific system, e.g. $\{|+\rangle,|-\rangle\}$ for a spin half particle.]

then the following results and definitions apply:

5. The set of all kets representing every possible physical state of a physical system forms a complex vector space (or Hilbert space[2]) $\mathcal{H}$ also known as the state space for the system. Moreover, every ket belonging to this state space represents a possible physical state of the system.

6. From the probability interpretation of item 3a, the set of states $\{|n\rangle; n = 1, 2, \ldots\}$ are such that the probability amplitudes $\langle n|m\rangle$ are given by

$$\begin{aligned}\langle n|m\rangle = \delta_{nm} &= 1 \quad n = m \\ &= 0 \quad n \neq m\end{aligned}$$

[1] Or properties, provided the information can be obtained without mutual interference.

[2] A Hilbert space is a particular kind of vector space in which the limits of Cauchy sequences of state vectors are required to have certain properties of importance in more advanced work, but not of any significance here. The general term Hilbert space will, however, be used here.

These states form a set of orthonormal basis vectors (or basis states) for the state space of the system, i.e. every linear superposition of two or more state vectors $|1\rangle$, $|2\rangle$, $|3\rangle$, ..., is also a state of the quantum system e.g. the state $|\psi\rangle$ given by

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle + \dots$$

is a state of the system for all complex numbers $c_1$, $c_2$, $c_3$, .... The number of basis states defines the *dimension* of the Hilbert space.

7. For any complex number $c$, the ket vectors $|\psi\rangle$ and $c|\psi\rangle$ represent the *same* physical state of the system.

8. The probability amplitude $\langle\phi|\psi\rangle$ is identified as the *inner product* of the state vectors $|\phi\rangle$ and $|\psi\rangle$, so that $\mathcal{H}$ is an inner product space.

9. $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$ i.e. if the role of initial and final state is reversed, then the associated probability amplitude is replaced by its complex conjugate.

10. The closure relation:

$$\langle\phi|\psi\rangle = \sum_n \langle\phi|n\rangle\langle n|\psi\rangle$$

expresses the probability amplitude of finding the system in a state $\langle\phi|$ given that it was in a state $|\psi\rangle$, as a sum over the probability amplitudes of the system ending up in the state $\langle\phi|$ after passing through any one of the intermediate states $|n\rangle$. It is also sometimes called the completeness relation.

11. The completeness relations:

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle \qquad \langle\phi| = \sum_n \langle\phi|n\rangle\langle n|$$

These relationships show that *any* state of the system, whether expressed as a bra or ket vector, can be expressed as a linear combination or linear superposition of the set of intermediate states, $\{|n\rangle\}$. Either of these relationships can be 'derived' from the closure relation above by 'cancelling' either the bra $\langle\phi|$ or the ket $|\psi\rangle$.

The idea behind the use of the word 'complete' is that the intermediate states represent *all* the possible intermediate states (for a particular experimental arrangement), i.e. there are no intermediate states left out. The terminology is then to say that the set of states $\{|n\rangle; n = 1, 2, \dots\}$ are *complete*. The reason for this terminology is best appreciated if we look at the next condition.

12. Normalization condition:

$$\langle\psi|\psi\rangle = \sum_n \langle\psi|n\rangle\langle n|\psi\rangle = \sum_n |\langle n|\psi\rangle|^2 = 1.$$

This result tells us that the probability of finding the system in any of the final states $\langle n|$ adds up to unity. Moreover, we know that $\langle n|m\rangle = 0$ if $n \neq m$, so that a system in the state $|n\rangle$ can never be found in some other state $|m\rangle$. Consequently, the set of basis states $\{|n\rangle; n = 1, 2, \dots\}$ represents a complete set of possible alternative final states, complete in the sense that the total probability of all the mutually exclusive possible final states is unity.