MACQUARIE UNIVERSITY

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Division of ICS

Sample Problems on Quantum Dynamics for PHYS301

1. The negative oxygen molecule ion O_2^- consists of a pair of oxygen atoms separated by a distance $2a$. As an approximation, the electron can be assumed to be found only on one or the other of the oxygen atoms, so that the eigenvalues for the position operator $\widehat{x}$ for the electron can be taken to be $-a$ and $+a$ and the corresponding eigenstates are $|-a\rangle$ and $|a\rangle$. With respect to these position eigenstates as basis vectors, the column vectors and matrix representing the states $|\pm a\rangle$ and the Hamiltonian $\widehat{H}$ for the electron can be written respectively as

$$|a\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix} \qquad |-a\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix} \text{ and } \qquad \widehat{H} \doteq \begin{pmatrix}0 & -A\\ -A & 0\end{pmatrix}$$

where A is a real number.

(a) Confirm, by direct substitution, that the eigenstates and energy eigenvalues of $\widehat{H}$ are given by

$$|1\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}; \quad E_1 = -A \quad \text{and} \quad |2\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}; \quad E_2 = A$$

(b) The electron is initially placed on the oxygen atom at $x = -a$. By expanding the state vector $|\psi(t)\rangle$ in the basis $\{|1\rangle, |2\rangle\}$ and using the postulate that time dependence of quantum mechanical amplitudes $\langle j|\psi(t)\rangle$ are given by $e^{-iE_jt/\hbar}\langle j|\psi(0)\rangle$ where $\widehat{H}|j\rangle = E_j|j\rangle$, show that

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{iAt/\hbar}|1\rangle - e^{-iAt/\hbar}|2\rangle\right]$$

(c) What is the probability of observing the system in state $|-a\rangle$ at a later time t, and provide a physical interpretation for your result.

(d) Show that $\widehat{x}|1\rangle = a|2\rangle$ and $\widehat{x}|2\rangle = a|1\rangle$, and hence determine the matrix representing $\widehat{x}$ in the $\{|1\rangle, |2\rangle\}$ basis.

(e) Calculate the expectation value of the position of the electron as a function of time when in the state $|\psi(t)\rangle$ above.

SOLUTION

(a) $$\widehat{H}|1\rangle \doteq \begin{pmatrix}0 & -A\\ -A & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = -A\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = -A|1\rangle.$$

$$\widehat{H}|2\rangle \doteq \begin{pmatrix}0 & -A\\ -A & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = A\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = A|2\rangle.$$

(b) Expanding the state $|\psi(t)\rangle$ in the $\{|1\rangle, |2\rangle\}$ basis gives

$$|\psi(t)\rangle = |1\rangle\langle 1|\psi(t)\rangle + |2\rangle\langle 2|\psi(t)\rangle.$$

Making use of the postulate concerning the time dependence of quantum mechanical amplitudes, this becomes

$$\begin{aligned}|\psi(t)\rangle &= e^{-iE_1t/\hbar}|1\rangle\langle 1|\psi(0)\rangle + e^{iE_2t/\hbar}|2\rangle\langle 2|\psi(0)\rangle \\ &= e^{iAt/\hbar}|1\rangle\langle 1|\psi(0)\rangle + e^{-iAt/\hbar}|2\rangle\langle 2|\psi(0)\rangle.\end{aligned}$$

We therefore need to calculate the probability amplitudes $\langle 1|\psi(0)\rangle$ and $\langle 2|\psi(0)\rangle$. Making use of the fact that

$$|\psi(0)\rangle = |-a\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$$

we therefore get

$$\begin{aligned}\langle 1|\psi(0)\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}} \\ \langle 2|\psi(0)\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{\sqrt{2}}.\end{aligned}$$

and hence

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\Big[e^{iAt/\hbar}|1\rangle - e^{-iAt/\hbar}|2\rangle\Big]$$

(c) The probability of finding the system in the state $|-a\rangle$ at a time t is given by $|\langle -a|\psi(t)\rangle|^2$. Calculating the probability amplitude $\langle -a|\psi(t)\rangle$

$$\langle -a|\psi(t)\rangle = \frac{1}{\sqrt{2}}\Big[e^{iAt/\hbar}\langle -a|1\rangle - e^{-iAt/\hbar}\langle -a|2\rangle\Big]$$

requires evaluating the probability amplitudes $\langle -a|1\rangle$ and $\langle -a|2\rangle$:

$$\begin{aligned}\langle -a|1\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}} \\ \langle -a|2\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = -\frac{1}{\sqrt{2}}\end{aligned}$$

and hence

$$\langle -a|\psi(t)\rangle = \frac{1}{2}\Big[e^{iAt/\hbar} + e^{-iAt/\hbar}\Big] = \cos\big(At/\hbar\big).$$

Thus we have the required probability

$$|\langle -a|\psi(t)\rangle|^2 = \cos^2\big(At/\hbar\big).$$

This probability oscillates with a period of $T = \pi\hbar/A$. Initially, it is unity, implying that there is 100% probability of finding the atom in the state $|-a\rangle$, i.e. the electron is on the oxygen atom positioned at $x = -a$. After a time $t = T/2$, this probability reduces to zero. The electron is then to be found with 100% certainty on the other oxygen atom.

(d) **NOTE: In the following set of solutions to this problem, we will be using a different basis set, that is $\{|1\rangle, |2\rangle\}$ as compared to the preceding parts of this example where the basis set $\{|a\rangle, |-a\rangle\}$ was used. Care should be taken in comparing the column vectors and matrices obtained here with those obtained earlier as the components will now be with reference to these different basis vectors.**

In the $\{|1\rangle, |2\rangle\}$ basis, the position operator for the electron, $\widehat{x}$, will be represented by the matrix

$$\widehat{x} \doteq \begin{pmatrix} \langle 1|\widehat{x}|1\rangle & \langle 1|\widehat{x}|2\rangle \\ \langle 2|\widehat{x}|1\rangle & \langle 2|\widehat{x}|2\rangle \end{pmatrix}.$$

To evaluate these matrix elements, we need to determine the result of $\widehat{x}$ acting on the state vectors $|1\rangle$ and $|2\rangle$. We do this by expanding these state vectors in the $\{|-a\rangle, |a\rangle\}$ basis, i.e.

$$|1\rangle = |-a\rangle\langle -a|1\rangle + |a\rangle\langle a|1\rangle.$$

We need the probability amplitudes $\langle -a|1\rangle$ and $\langle a|1\rangle$:

$$\begin{aligned} \langle -a|1\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \\ \langle a|1\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}. \end{aligned}$$

Thus

$$|1\rangle = \frac{1}{\sqrt{2}}\big[|-a\rangle + |a\rangle\big]$$

and hence

$$\begin{aligned} \widehat{x}|1\rangle &= \frac{1}{\sqrt{2}}\big[\widehat{x}|-a\rangle + \widehat{x}|a\rangle\big] \\ &= \frac{1}{\sqrt{2}}\big[-a|-a\rangle + a|a\rangle\big] \\ &\doteq a\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Thus, $\widehat{x}|1\rangle = a|2\rangle$. In a similar way we have for $|2\rangle$:

$$|2\rangle = |-a\rangle\langle -a|2\rangle + |a\rangle\langle a|2\rangle.$$

The probability amplitudes $\langle -a|2\rangle$ and $\langle a|2\rangle$ will be

$$\begin{aligned} \langle -a|2\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{1}{\sqrt{2}} \\ \langle a|2\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \end{aligned}$$

so that

$$|2\rangle = -\frac{1}{\sqrt{2}}\big[|-a\rangle - |a\rangle\big]$$

and hence

$$\begin{aligned} \widehat{x}|2\rangle &= -\frac{1}{\sqrt{2}}\big[\widehat{x}|-a\rangle - \widehat{x}|a\rangle\big] \\ &= -\frac{1}{\sqrt{2}}\big[-a|-a\rangle - a|a\rangle\big] \\ &\doteq a\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus, $\widehat{x}|2\rangle = a|1\rangle$.
From these results, it immediately follows that

$$\widehat{x} \doteq \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

(e) The expectation value of the position of the electron as a function of time is given by $\langle x(t)\rangle = \langle\psi(t)|\widehat{x}|\psi(t)\rangle$. Using the representations of the vectors and the operator in the $\{|1\rangle, |2\rangle\}$ basis, we have

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{iAt/\hbar} \\ -e^{iAt/\hbar} \end{pmatrix}.$$

The corresponding bra will be

$$\langle\psi(t)| \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-iAt/\hbar} & e^{iAt/\hbar} \end{pmatrix}$$

and hence

$$\begin{aligned}\langle x(t)\rangle &= \tfrac{1}{2}\begin{pmatrix} e^{-iAt/\hbar} & e^{iAt/\hbar} \end{pmatrix}\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} e^{iAt/\hbar} \\ -e^{iAt/\hbar} \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} ae^{iAt/\hbar} & ae^{-iAt/\hbar} \end{pmatrix}\begin{pmatrix} e^{iAt/\hbar} \\ -e^{-iAt/\hbar} \end{pmatrix} \\ &= -a\cos\left(2At/\hbar\right).\end{aligned}$$

Thus, at $t = 0$, the average position of the electron is at $x = -a$ which is consistent with the fact that the electroon was initially positioned at $x = -a$. The average position oscillates with a period of $T = \pi\hbar/A$ and with an amplitude of a, i.e. the average position swings between the two extremes of $x = \pm a$, corresponding to the positions of the two oxygen atoms.

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2. A carbon dioxide molecule, O = C = O, where the two oxygen atoms are positioned equal distances either side of the carbon atom is negatively charged by acquiring an electron which can attach itself to any one of the atoms. Assuming that the carbon atom is positioned at $x = 0$, and the oxygen atoms are positioned a distance a on either side of the carbon atom, then the possible positions of the electron are $x = -a$, $x = 0$, and $x = +a$. The electron can tunnel through the finite potential barrier that exists between the carbon atom and each oxygen atom.

Using the position eigenstates as basis vectors, the Hamiltonian for the system is

$$\widehat{H} \doteq \begin{pmatrix} \langle -a|\widehat{H}|-a\rangle & \langle -a|\widehat{H}|0\rangle & \langle -a|\widehat{H}|a\rangle \\ \langle 0|\widehat{H}|-a\rangle & \langle 0|\widehat{H}|0\rangle & \langle 0|\widehat{H}|a\rangle \\ \langle a|\widehat{H}|-a\rangle & \langle a|\widehat{H}|0\rangle & \langle a|\widehat{H}|a\rangle \end{pmatrix} = \begin{pmatrix} E_0 & -V & 0 \\ -V & E_0 & -V \\ 0 & -V & E_0 \end{pmatrix}$$

where V is a real quantity. The energy eigenstates of $\widehat{H}$ can be shown to be

$$\begin{aligned}|\mathrm{I}\rangle &= \tfrac{1}{\sqrt{2}}|-a\rangle - \tfrac{1}{\sqrt{2}}|a\rangle \\ |\mathrm{II}\rangle &= \tfrac{1}{2}|-a\rangle - \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{2}|a\rangle \\ |\mathrm{III}\rangle &= \tfrac{1}{2}|-a\rangle + \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{2}|a\rangle\end{aligned}$$

and their corresponding eigenvalues are $E_{\mathrm{I}} = E_0$, $E_{\mathrm{II}} = E_0 + \sqrt{2}V$, and $E_{\mathrm{III}} = E_0 - \sqrt{2}V$.

(a) (i) Write down the eigenvalue equation for the position operator $\widehat{x}$ for the electron.

(ii) What are the stationary states for this system? Justify your answer by calculating the expectation value of the position operator $\widehat{x}$ for the system in one of these stationary states.

(iii) Why are the position eigenstates *not* stationary states? Hence give a physical meaning to the off-diagonal elements $-V$.

(b) At $t = 0$ the electron is observed to be on the carbon atom.

(i) Obtain an expression for the state $|\psi(t)\rangle$ at some later time t.

(ii) What is the probability of the electron still being observed on the carbon atom at time t?

(iii) What does this last result mean physically?

SOLUTION

(a) (i) The eigenvalue equation is $\widehat{x}|x\rangle = x|x\rangle$, where the eigenvalues are $x = 0$ and $\pm a$ and the associated eigenstates are $|0\rangle$, and $|\pm a\rangle$.

(ii) The stationary states are the states $|\mathrm{I}\rangle$, $|\mathrm{II}\rangle$ and $|\mathrm{III}\rangle$. These states are eigenstates of the Hamiltonian, and hence are stationary states in that the expectation values of any observable of the system is independent of time if the system is initially placed in one of these states.
We can illustrate this by supposing that the system is initially in the state $|\psi(0)\rangle = |\mathrm{I}\rangle$. Then

$$|\psi(t)\rangle = |\mathrm{I}\rangle e^{-iE_{\mathrm{I}}t/\hbar}.$$

From this, the corresponding bra vector is

$$\langle\psi(t)| = e^{iE_{\mathrm{I}}t/\hbar}\langle\mathrm{I}|.$$

Overall then

$$< x(t) >= \langle\psi(t)|\widehat{x}|\psi(t)\rangle = e^{iE_{\mathrm{I}}t/\hbar}\langle\mathrm{I}|\widehat{x}|\mathrm{I}\rangle e^{-iE_{\mathrm{I}}t/\hbar} = \langle\mathrm{I}|\widehat{x}|\mathrm{I}\rangle$$

which is independent of time, as required.

(iii) The position eigenstates would be eigenstates of the Hamiltonian if $V = 0$ as the Hamiltonian matrix would then be diagonal. The position eigenstates would then be stationary states which means that if the electron was initially placed in a position eigenstate, it would remain there, implying that there is an infinitely high potential barrier between the atoms that prevents the electron moving from one atom to another. However, if $V \neq 0$, then the position eigenstates are *not* eigenstates of the Hamiltonian and hence are not stationary states. In other words, if the electron is initially placed in any of the position eigenstates, it will not remain in that state. Thus $V \neq 0$ means that there is at most a finite potential barrier between the atoms.

(b) (i) Let $|\psi(t)\rangle$ be the state of the electron at time t. Initially it is known to be on the carbon atom, so the initial state is $|\psi(0)\rangle = |0\rangle$. The state $|\psi(t)\rangle$ can be expanded in terms of the basis states $|\mathrm{I}\rangle$, $|\mathrm{II}\rangle$ and $|\mathrm{III}\rangle$:

$$\begin{aligned}|\psi(t)\rangle =& |\mathrm{I}\rangle\langle\mathrm{I}|\psi(t)\rangle + |\mathrm{II}\rangle\langle\mathrm{II}|\psi(t)\rangle + |\mathrm{III}\rangle\langle\mathrm{III}|\psi(t)\rangle \\ =& |\mathrm{I}\rangle\langle\mathrm{I}|\psi(0)\rangle e^{-iE_{\mathrm{I}}t/\hbar} + |\mathrm{II}\rangle\langle\mathrm{II}|\psi(0)\rangle e^{-iE_{\mathrm{II}}t/\hbar} + |\mathrm{III}\rangle\langle\mathrm{III}|\psi(0)\rangle e^{-iE_{\mathrm{III}}t/\hbar} \\ =& |\mathrm{I}\rangle\langle\mathrm{I}|0\rangle e^{-iE_{\mathrm{I}}t/\hbar} + |\mathrm{II}\rangle\langle\mathrm{II}|0\rangle e^{-iE_{\mathrm{II}}t/\hbar} + |\mathrm{III}\rangle\langle\mathrm{III}|0\rangle e^{-iE_{\mathrm{III}}t/\hbar}\end{aligned}$$

What is now required are the inner products $\langle \mathrm{I}|0\rangle$, $\langle \mathrm{II}|0\rangle$ and $\langle \mathrm{III}|0\rangle$. Using the given expressions for $|\mathrm{I}\rangle$, $|\mathrm{II}\rangle$ and $|\mathrm{III}\rangle$, the corresponding bra vectors are

$$\begin{aligned}
\langle \mathrm{I}| &= \tfrac{1}{\sqrt{2}}\langle -a| - \tfrac{1}{\sqrt{2}}\langle a| \\
\langle \mathrm{II}| &= \tfrac{1}{2}\langle -a| - \tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{2}\langle a| \\
\langle \mathrm{III}| &= \tfrac{1}{2}\langle -a| + \tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{2}\langle a|
\end{aligned}$$

and hence the inner products are

$$\langle \mathrm{I}|0\rangle = 0 \quad \langle \mathrm{II}|0\rangle = -\tfrac{1}{\sqrt{2}} \quad \langle \mathrm{III}|0\rangle = \tfrac{1}{\sqrt{2}}.$$

Thus, after substituting for $E_{\mathrm{II}} = E_0 + \sqrt{2}V$ and $E_{\mathrm{III}} = E_0 - \sqrt{2}V$, the time dependent state is

$$\begin{aligned}
|\psi(t)\rangle &= -\tfrac{1}{\sqrt{2}}|\mathrm{II}\rangle e^{-iE_{\mathrm{II}}t/\hbar} + \tfrac{1}{\sqrt{2}}e^{-iE_{\mathrm{III}}t/\hbar} \\
&= -\tfrac{1}{\sqrt{2}}e^{-iE_0t/\hbar}\left[e^{i\sqrt{2}Vt/\hbar}|\mathrm{III}\rangle - e^{-i\sqrt{2}Vt/\hbar}|\mathrm{II}\rangle\right].
\end{aligned}$$

(ii) The probability of the electron still being observed on the carbon atom at time t is given by $|\langle 0|\psi(t)\rangle|^2$. From the last result

$$\langle 0|\psi(t)\rangle = -\tfrac{1}{\sqrt{2}}e^{-iE_0t/\hbar}\left[e^{i\sqrt{2}Vt/\hbar}\langle 0|\mathrm{III}\rangle - e^{-i\sqrt{2}Vt/\hbar}\langle 0|\mathrm{II}\rangle\right]. \tag{1}$$

The inner products $\langle 0|\mathrm{III}\rangle$ and $\langle 0|\mathrm{II}\rangle$ follow directly from $\langle \mathrm{II}|0\rangle$ and $\langle \mathrm{III}|0\rangle$ evaluated above, so that

$$\langle 0|\psi(t)\rangle = -\tfrac{1}{2}e^{-iE_0t/\hbar}\left[e^{i\sqrt{2}Vt/\hbar} + e^{-i\sqrt{2}Vt/\hbar}\right] = -e^{-iE_0t/\hbar}\cos(\sqrt{2}Vt/\hbar)$$

and hence

$$|\langle 0|\psi(t)\rangle|^2 = \cos^2(\sqrt{2}Vt/\hbar) = \tfrac{1}{2}\left(1 + \cos(2\sqrt{V}t/\hbar)\right).$$

(iii) This result means that the probability of the electron being found on the carbon oscillates in time with an angular frequency $\omega = 2\sqrt{V}/\hbar$. The probability of the electron being found on the carbon atom is initially unity, but will be zero after a time $t = \pi/\omega$, and unity again at a time $t = 2\pi/\omega$, and so on periodically after that.

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3. The ammonia molecule consists of a plane of hydrogen atoms arranged in an equilateral triangle, with the nitrogen atom positioned symmetrically either above or below this plane, thereby forming a triangular pyramid shape. If we let $|1\rangle$ and $|2\rangle$ be the position eigenstates for the nitrogen atom, corresponding to the atom being either above or below the plane of hydrogen atoms respectively, then the Hamiltonian matrix for the molecule can be shown to be, in the $\{|1\rangle, |2\rangle\}$ basis

$$\hat{H} \doteq \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}.$$

(a) Assuming that the state of the system at time t can be expressed as

$$|\psi(t)\rangle = C_1(t)|1\rangle + C_2(t)|2\rangle,$$

write down the Schrödinger equation for this system in matrix form.

(b) Confirm, by direct substitution into the equations for $C_1(t)$ and $C_2(t)$ that the solutions for these coefficients are

$$\begin{aligned} C_1(t) &= \tfrac{1}{2}e^{-iE_0t/\hbar}\left(ae^{iAt/\hbar} + be^{-iAt/\hbar}\right) \\ C_2(t) &= \tfrac{1}{2}e^{-iE_0t/\hbar}\left(ae^{iAt/\hbar} - be^{-iAt/\hbar}\right) \end{aligned}$$

where a and b are unknown constants.

(c) The system is initially in the state $|1\rangle$. Solve for the probability of observing the system in state $|2\rangle$ at a later time t, and provide a physical interpretation for your result.

SOLUTION

(a) In the $\{|1\rangle, |2\rangle\}$ basis, the state vector $|\psi(t)\rangle$ will be represented by the column vector:

$$|\psi(t)\rangle \doteq \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}$$

so that the Schrödinger equation $\widehat{H}|\psi(t)\rangle = i\hbar\dfrac{d}{dt}|\psi(t)\rangle$ will be

$$\begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}\begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} = i\hbar\begin{pmatrix} \dot{C}_1(t) \\ \dot{C}_2(t). \end{pmatrix}$$

(b) The matrix equation for the coefficients C_1 and C_2 can be written as two coupled equations:

$$\begin{aligned} i\hbar\dot{C}_1 &= E_0C_1 - AC_2 \\ i\hbar\dot{C}_2 &= -AC_1 + E_0C_2. \end{aligned}$$

Taking the derivative with respect to time of the expression for C_1 and C_2 gives:

$$\begin{aligned} \dot{C}_1(t) =& \tfrac{1}{2}\big(-iE_0/\hbar\big)e^{-iE_0t/\hbar}\left(ae^{iAt/\hbar} + be^{-iAt/\hbar}\right) \\ &+ \tfrac{1}{2}e^{-iE_0t/\hbar}\left[a\big(iA/\hbar\big)e^{iAt/\hbar} + b\big(-iA/\hbar\big)e^{-iAt/\hbar}\right] \\ =& \big(-iE_0/\hbar\big)C_1(t) + \big(iA/\hbar\big)C_2(t) \end{aligned}$$

and

$$\begin{aligned}\dot{C}_2(t) &= \tfrac{1}{2}\big(-iE_0/\hbar\big)e^{-iE_0t/\hbar}\left(ae^{iAt/\hbar} - be^{-iAt/\hbar}\right)\\ &\quad + \tfrac{1}{2}e^{-iE_0t/\hbar}\Big[a\big(iA/\hbar\big)e^{iAt/\hbar} - b\big(-iA/\hbar\big)e^{-iAt/\hbar}\Big]\\ &= \big(-iE_0/\hbar\big)C_2(t) + \big(iA/\hbar\big)C_1(t).\end{aligned}$$

Multiplying through by $i\hbar$ then gives the correct pair of coupled equations for C_1 and C_2.

(c) If the system was intially in the state $|1\rangle$, then $C_1(0) = 1$ and $C_2(0) = 0$. Thus we get the pair of equations for a and b:

$$\begin{aligned}1 &= \tfrac{1}{2}\big(a+b\big)\\ 0 &= a - b\end{aligned}$$

so that $a = b = 1$. Consequently

$$\begin{aligned}C_1(t) &= e^{-iE_0t/\hbar}\left(e^{iAt/\hbar} + e^{-iAt/\hbar}\right) = \tfrac{1}{2}e^{-iE_0t/\hbar}\cos\big(At/\hbar\big)\\ C_2(t) &= \tfrac{1}{2}e^{-iE_0t/\hbar}\left(e^{iAt/\hbar} - e^{-iAt/\hbar}\right) = ie^{-iE_0t/\hbar}\sin\big(At/\hbar\big)\end{aligned}$$

The probability amplitude of finding the system in the state $|2\rangle$ at time t is then

$$\langle 2|\psi(t)\rangle = C_2(t) = ie^{-iE_0t/\hbar}\sin\big(At/\hbar\big).$$

Thus, the probability of finding the system in the state $|2\rangle$ will be

$$|\langle 2|\psi(t)\rangle|^2 = \sin^2\big(At/\hbar\big).$$

This result tells us that the probability of finding the system in the state $|2\rangle$ is initially zero, as it should be since at that tiome the system was prepared in the state $|1\rangle$. But as time advances, the probability of finding the system in the state $|2\rangle$ will increase, becoming unity at a time $\pi\hbar/A$, and so on periodically after that.
