MACQUARIE UNIVERSITY

Department of Physics
Division of ICS

Sample Problems on State Vectors for PHYS301

1. A spin half particle is prepared in the spin state

$$|S\rangle = 2|+\rangle + (1+\sqrt{3}i)|-\rangle$$

What is the probability of finding in the spin up state (i.e. in the state $|+\rangle$, for which $S_z = \frac{1}{2}\hbar$)?

SOLUTION

We should first check to see that this state is normalized to unity. The bra vector corresponding to this state is

$$\langle S| = 2\langle +| + (1-\sqrt{3}i)\langle -|$$

so that the inner product $\langle S|S\rangle$ is given by

$$\begin{aligned}\langle S|S\rangle &= \Big(2\langle +| + (1-\sqrt{3}i)\langle -|\Big)\Big(2|+\rangle + (1+\sqrt{3}i)|-\rangle\Big)\\ &= 4\langle +|+\rangle + 2(1+\sqrt{3}i)\langle +|-\rangle + 2(1-\sqrt{3}i)\langle -|+\rangle\\ &\quad + (1-\sqrt{3}i)(1+\sqrt{3}i)\langle -|-\rangle.\end{aligned}$$

Using the fact that $\{|+\rangle, |-\rangle\}$ form an orthonormal basis for the state space for the spin of the particle, we have $\langle +|+\rangle = \langle -|-\rangle = 1$ and $\langle -|+\rangle = \langle +|-\rangle = 0$ so that

$$\langle S|S\rangle = 8.$$

This state is not normalized to unity, so in order to calculate probabilities correctly, we must renormalize $|S\rangle$. We do this by defining

$$|\widetilde{S}\rangle = \frac{|S\rangle}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\Big(2|+\rangle + (1+\sqrt{3}i)|-\rangle\Big)$$

which now satisfies $\langle\widetilde{S}|\widetilde{S}\rangle = 1$. But note that the new state vector $|\widetilde{S}\rangle$ still represents the *same physical state* as the original vector $|S\rangle$.

We can now calculate the probability of finding the particle to have spin $S_z = \frac{1}{2}\hbar$ by calculating

$$|\langle +|\widetilde{S}\rangle|^2 = |2/2\sqrt{2}|^2 = \tfrac{1}{2}.$$

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2. In so-called isospin theory used to describe the strong force between nuclear particles, the neutron and the proton are assumed to be two different states of the one particle called the nucleon. We can represent the state in which the nucleon is a neutron by $|n\rangle$, and the state in which it is a proton by $|p\rangle$. Suppose the nucleon is prepared in the state

$$|\psi\rangle = \frac{i}{\sqrt{3}}|p\rangle + \sqrt{\frac{1}{3}}(1+i)|n\rangle$$

determine the probability amplitude of the particle being observed to be a proton or a neutron, and the corresponding probabilities for the particle to be observed to be a proton or a neutron. Is the state correctly normalized?

SOLUTION

The probability amplitude for being observed to be a proton will be

$$\langle p|\psi\rangle = \frac{i}{\sqrt{3}}$$

and the corresponding probability will be

$$|\langle p|\psi\rangle|^2 = \left|\frac{i}{\sqrt{3}}\right|^2 = \frac{1}{3}.$$

The probability amplitude for being observed to be a neutron will be

$$\langle n|\psi\rangle = \sqrt{\frac{2}{3}}(1+i)$$

and the corresponding probability will be

$$|\langle n|\psi\rangle|^2 = \left|\sqrt{\frac{1}{3}}(1+i)\right|^2 = \frac{2}{3}.$$

Since $|\langle p|\psi\rangle|^2 + |\langle n|\psi\rangle|^2 = 1$, the state $|\psi\rangle$ is normalized to unity.

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3. The state vector for a spin half particle that passes through a magnetic field oriented in the direction $\widehat{\mathbf{n}}$ and exits with its spin component in the direction of the magnetic field, i.e. $S = \mathbf{S} \cdot \widehat{\mathbf{n}} = \frac{1}{2}\hbar$ is given by

$$|S\rangle = \cos\tfrac{1}{2}\theta\,|+\rangle + \sin\tfrac{1}{2}\theta\,e^{i\phi}|-\rangle$$

where $\widehat{\mathbf{n}} = \sin\theta\cos\phi\widehat{\mathbf{i}} + \sin\theta\sin\phi\widehat{\mathbf{j}} + \cos\theta\widehat{\mathbf{k}}$. This is the most general ket vector for a spin half system.

(a) What is the corresponding bra vector?
(b) Show that this state is normalized to unity.
(c) Identify the state $|S\rangle$ if $\widehat{\mathbf{n}} = \widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, and $\widehat{\mathbf{k}}$, and express $|S\rangle$ in terms of the basis states $\{|-\rangle, |+\rangle\}$ in each case.
(d) Show that the state $|-S\rangle$ given by making the substitutions $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$ (which means that $\widehat{\mathbf{n}} \to -\widehat{\mathbf{n}}$) is orthogonal to $|S\rangle$.

SOLUTION

(a) The bra vector correponding to $|S\rangle$ is $\langle S| = \cos\frac{1}{2}\theta\,\langle +| + \sin\frac{1}{2}\theta\,e^{-i\phi}\langle -|$.

(b) $$\begin{aligned}\langle S|S\rangle =&\Big(\cos\tfrac{1}{2}\theta\,\langle +| + \sin\tfrac{1}{2}\theta\,e^{-i\phi}\langle -|\Big)\Big(\cos\tfrac{1}{2}\theta\,|+\rangle + \sin\tfrac{1}{2}\theta\,e^{i\phi}|-\rangle\Big)\\ =&\cos^2\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\theta\,e^{-i\phi}e^{i\phi}\\ =&\cos^2\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\theta\\ =&1.\end{aligned}$$

(c) If $\widehat{\mathbf{n}} = \widehat{\mathbf{i}}$, then $S = S_x$, so the state $|S\rangle$ is just the state $|S_x = \frac{1}{2}\hbar\rangle$. In this case, $\theta = \frac{1}{2}\pi$, and $\phi = 0$ so that

$$|S_x = \tfrac{1}{2}\hbar\rangle = \cos\tfrac{1}{4}\pi\,|+\rangle + \sin\tfrac{1}{4}\pi\,|-\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle + |-\rangle\Big)$$

If $\widehat{\mathbf{n}} = \widehat{\mathbf{j}}$, then $S = S_y$, so the state $|S\rangle$ is just the state $|S_y = \frac{1}{2}\hbar\rangle$. In this case, $\theta = \frac{1}{2}\pi$ and $\phi = \frac{1}{2}\pi$ so that

$$|S_y = \tfrac{1}{2}\hbar\rangle = \cos\tfrac{1}{4}\pi\,|+\rangle + \sin\tfrac{1}{4}\pi\,e^{i\pi/2}\,|-\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle + i|-\rangle\Big)$$

Finally, if $\widehat{\mathbf{n}} = \widehat{\mathbf{k}}$, then $S = S_z$, so the state $|S\rangle$ is just the state $|S_z = \frac{1}{2}\hbar\rangle$. In this case, $\theta = 0$ and ϕ can have any value, so that

$$|S_z = \tfrac{1}{2}\hbar\rangle = \cos 0\,|+\rangle + \sin 0\,|-\rangle = |+\rangle$$

as expected.

(d) The new state is

$$\begin{aligned}|-S\rangle &= \cos\tfrac{1}{2}(\pi-\theta)|+\rangle + \sin\tfrac{1}{2}(\pi-\theta)\,e^{i(\phi+\pi)}|-\rangle\\ &= \sin\tfrac{1}{2}\theta\,|+\rangle - \cos\tfrac{1}{2}\theta\,e^{i\phi}|-\rangle.\end{aligned}$$

We want to check that this state is orthogonal to $|S\rangle$, thus we have to calculate $\langle -S|S\rangle$:

$$\begin{aligned}\langle -S|S\rangle =&\Big(\sin\tfrac{1}{2}\theta\langle +| - \cos\tfrac{1}{2}\theta\,e^{-i\phi}\langle -|\Big)\Big(\cos\tfrac{1}{2}\theta\,|+\rangle + \sin\tfrac{1}{2}\theta\,e^{i\phi}|-\rangle\Big)\\ =&\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - \cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta\\ =&0.\end{aligned}$$

4. An ozone molecule consists of three oxygen atoms lying at the corners of an equilateral triangle. The molecule can be ionized by the addition of an electron which can reside on any one of the three atoms. If we name the atoms 1, 2, and 3 respectively, then we can represent the three possible states of the electron by the kets $|1\rangle$, $|2\rangle$ and $|3\rangle$, where the state $|n\rangle$ means that the electron is residing on atom 1.

   (a) What is the dimension of the state space for the electron?

   (b) Construct a state in which the electron has equal probabilities of residing on either atom 1 or atom 2, but has zero probability of being found on atom 3. Check that your state is normalized to unity.

   (c) Construct a state that is orthogonal to the state constructed in the previous question.

SOLUTION

(a) As there are three basis states, the state space has dimension 3.

(b) We want a state of the form $|\psi\rangle = a|1\rangle + b|2\rangle + 0|3\rangle = a|1\rangle + b|2\rangle$. Note that the coefficeint of $|3\rangle$ is taken to be zero, guaranteeing that there is no probability of finding the electron on atom 3.

We also require a and b such that $|a|^2 = |b|^2$, i.e. equal probabilities of finding the particles on either atom 1 or 2 and further require that $|a|^2 + |b|^2 = 1$ so that the state is correctly normalized to unity.

Thus we must have $|a|^2 = |b|^2 = \frac{1}{2}$ and hence

$$a = \frac{1}{\sqrt{2}}e^{i\phi} \qquad b = \frac{1}{\sqrt{2}}e^{i\eta}$$

where ϕ and η are phase factors that we do not have enough information to determine. So we choose the simplest possibility, that is $\phi = \eta = 0$ to give

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|1\rangle + |2\rangle\Big)$$

though any other choice for ϕ and η yields a perfectly correct result.

(c) The most obvious state is $|3\rangle$ i.e. we immediately have that $\langle 3|\psi\rangle = 0$. But other states are possible, such as

$$|\xi\rangle = \frac{1}{\sqrt{2}}\Big(|1\rangle - |2\rangle\Big).$$

There are an infinite number of choices.

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5. A particle in an infinite potential well of width L can have the energies $E_n = \pi^2\hbar^2 n^2/2mL^2$ where $n = 1, 2, \ldots$. We can specify the states of the particle in the well by the kets $|1\rangle$, $|2\rangle$, $|3\rangle$,... where $|n\rangle$ is the ket corresponding to the particle having the energy E_n. These states form a complete orthonormal set of basis states for the particle in the well.

(a) What is the dimension of the state space for the particle?

(b) State the orthonormality conditions for the kets $\{|1\rangle, |2\rangle, \ldots\}$.

(c) A particle is prepared in the state

$$|\psi\rangle = \tfrac{1}{3}|1\rangle + \tfrac{1}{3}(2+i)|2\rangle + \alpha|3\rangle.$$

This state is normalized to unity. If the experiment is repeated 1000 times under identical conditions, and the energy of the particle in the well is measured, roughly how many times will the particle be observed to have the energy E_3?

SOLUTION

(a) Since the number of basis states is infinite, the state space has dimension of infinity.

(b) The orthonormality condition can be stated as

$$\begin{aligned}\langle m|n\rangle = \delta_{mn} &= 1 \text{ if } n = m \\ &= 0 \text{ if } n \neq m.\end{aligned}$$

(c) Since the state $|\psi\rangle$ is normalized to unity, we must have $\langle\psi|\psi\rangle = 1$, i.e.

$$\begin{aligned}\langle\psi|\psi\rangle =& \Big(\tfrac{1}{3}\langle 1| + \tfrac{1}{3}(2-i)\langle 2| + \alpha^*\langle 3|\Big)\Big(\tfrac{1}{3}|1\rangle + \tfrac{1}{3}(2+i)|2\rangle + \alpha|3\rangle\Big) \\ =& \tfrac{2}{3} + |\alpha|^2 = 1\end{aligned}$$

and hence $|\alpha|^2 = \frac{1}{3}$, i.e. the probability of finding the particle in the state $|3\rangle$ is $\frac{1}{3}$. Thus, if the experiment is repeated 1000 times, on roughly $\frac{1}{3} \times 1000 \approx 333$ occasions the energy of the particle will be measured to be E_3.
