MACQUARIE UNIVERSITY

Department of Physics
Division of ICS

Sample Problems on Representations of Vectors and Operators for PHYS301

1. Consider two vectors

$$|1\rangle = \frac{1}{\sqrt{2}}\big[|-\rangle - i|+\rangle\big] \qquad \text{and} \qquad |2\rangle = \frac{1}{\sqrt{2}}\big[|-\rangle + i|+\rangle\big]$$

where $|\pm\rangle$ are the usual base vectors for a spin half system, and the operator $\widehat{A}$ defined by

$$\widehat{A}|\pm\rangle = \pm\tfrac{1}{2}i\hbar|\mp\rangle.$$

Note: expressed as column vectors in the $\{|+\rangle, |-\rangle\}$ representation, the basis states are

$$|+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad |-\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}.$$

(a) Express the state vectors $|1\rangle$ and $|2\rangle$ as column vectors.

(b) Write down the corresponding bra vectors as row vectors.

(c) Calculate the inner products $\langle 1|2\rangle$ and $\langle 1|1\rangle$.

(d) Write this operator as a matrix in the $\{|+\rangle, |-\rangle\}$ representation.

(e) Calculate $\widehat{A}|1\rangle$ using the matrix representation of $\widehat{A}$ and the column representation of $|1\rangle$.

(f) Calculate $\langle 2|\widehat{A}$ using the matrix representation of $\widehat{A}$ and the row representation of $\langle 2|$.

SOLUTION

(a) The state vectors $|1\rangle$ and $|2\rangle$ are

$$|1\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\1\end{pmatrix} \qquad \text{and} \qquad |2\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix}.$$

(b) The corresponding bra vectors are then

$$\langle 1| \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}i & 1\end{pmatrix} \qquad \text{and} \qquad \langle 2| \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}-i & 1\end{pmatrix}.$$

(c) Calculating the inner products using these representations gives

$$\langle 1|2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}i & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix} = \frac{1}{2}(-1+1) = 0,$$

and, similarly

$$\langle 1|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}i & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}-i\\1\end{pmatrix} = \frac{1}{2}(1+1) = 1.$$

(d) Writing $\widehat{A}$ out as a matrix:

$$\begin{aligned}\widehat{A} &\doteq \begin{pmatrix} \langle +|\widehat{A}|+\rangle & \langle +|\widehat{A}|-\rangle \\ \langle -|\widehat{A}|+\rangle & \langle -|\widehat{A}|-\rangle \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\frac{1}{2}i\hbar \\ \frac{1}{2}i\hbar & 0 \end{pmatrix}.\end{aligned}$$

The manner in which each element of the matrix is evaluated is illustrated here for $\langle -|\widehat{A}|+\rangle$. Since $\widehat{A}|+\rangle = \frac{1}{2}i\hbar|-\rangle$, then

$$\langle -\widehat{A}|+\rangle \equiv \langle -|\big(\widehat{A}|+\rangle\big) = \langle -|\big(\tfrac{1}{2}i\hbar|+\rangle\big) = \tfrac{1}{2}i\hbar\langle +|+\rangle = \tfrac{1}{2}i\hbar,$$

and the other elements follow in the same fashion.

(e) The action of $\widehat{A}$ on $|1\rangle$, i.e. $\widehat{A}|1\rangle$ can then be evaluated as follows:

$$\begin{aligned}\widehat{A}|1\rangle &\doteq \begin{pmatrix} 0 & -\frac{1}{2}i\hbar \\ \frac{1}{2}i\hbar & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \\ &= -\tfrac{1}{2}i\hbar \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= \tfrac{1}{2}\hbar \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}\end{aligned}$$

i.e. $\widehat{A}|1\rangle = \frac{1}{2}\hbar|1\rangle$.

(f) Similarly, we can work out $\langle 2|\widehat{A}$:

$$\begin{aligned}\langle 2|\widehat{A} &\doteq \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2}i\hbar \\ \frac{1}{2}i\hbar & 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{2}i\hbar & -\frac{1}{2}\hbar \end{pmatrix} \\ &= -\tfrac{1}{2}\hbar \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \end{pmatrix}\end{aligned}$$

i.e. $\langle 2|\widehat{A} = -\frac{1}{2}\hbar\langle 2|$.

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2. With respect to a pair of orthonormal vectors $|\varphi_1\rangle$ and $|\varphi_2\rangle$ that span the Hilbert space $\mathcal{H}$ of a certain system, the operator $\hat{A}$ is defined by its action on these base states as follows

$$\hat{A}|\varphi_1\rangle = 2|\varphi_1\rangle + 2i|\varphi_2\rangle$$
$$\hat{A}|\varphi_2\rangle = 2i|\varphi_{1\rangle} - |\varphi_2\rangle.$$

(a) What is the matrix represention of $\hat{A}$ in the $\{|\varphi_1\rangle,|\varphi_2\rangle\}$ basis?

(b) For the state $|\psi\rangle = \Big(|\varphi_1\rangle + |\varphi_2\rangle\Big)/\sqrt{2}$, determine $\widehat{A}|\psi\rangle$ and $\langle\psi|\widehat{A}$ by using the representations of the vectors and operators in the $\{|\varphi_1\rangle, |\varphi_2\rangle\}$ basis. Write your result in bra ket notation.

(c) Is $\langle\psi|\widehat{A}$ the bra vector corresponding to $\widehat{A}|\psi\rangle$?

SOLUTION

(a) The matrix representing $\widehat{A}$ will be given by

$$\widehat{A} \doteq \begin{pmatrix} \langle\varphi_1|\widehat{A}|\varphi_1\rangle & \langle\varphi_1|\widehat{A}|\varphi_2\rangle \\ \langle\varphi_2|\widehat{A}|\varphi_1\rangle & \langle\varphi_2|\widehat{A}|\varphi_2\rangle \end{pmatrix}.$$

The various elements of this matrix are derived as follows:

$$\langle\varphi_1|\widehat{A}|\varphi_1\rangle \equiv \langle\varphi_1|\Big(\widehat{A}|\varphi_1\rangle\Big) = \langle\varphi_1|\Big(2|\varphi_1\rangle 2i|\varphi_2\rangle\Big) = 2$$
$$\langle\varphi_1|\widehat{A}|\varphi_2\rangle \equiv \langle\varphi_1|\Big(\widehat{A}|\varphi_2\rangle\Big) = \langle\varphi_1|\Big(2i|\varphi_{1\rangle} - |\varphi_2\rangle\Big) = 2i$$
$$\langle\varphi_2|\widehat{A}|\varphi_1\rangle \equiv \langle\varphi_2|\Big(\widehat{A}|\varphi_1\rangle\Big) = \langle\varphi_2|\Big(2|\varphi_1\rangle + 2i|\varphi_2\rangle\Big) = 2i$$
$$\langle\varphi_2|\widehat{A}|\varphi_2\rangle \equiv \langle\varphi_2|\Big(\widehat{A}|\varphi_2\rangle\Big) = \langle\varphi_2|\Big(2i|\varphi_{1\rangle} - |\varphi_2\rangle\Big) = -1$$

and hence

$$\widehat{A} \doteq \begin{pmatrix} 2 & 2i \\ 2i & -1 \end{pmatrix}.$$

(b) The column vector representing $|\psi\rangle$ will be

$$|\psi\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that we have

$$\widehat{A}|\psi\rangle \doteq \begin{pmatrix} 2 & 2i \\ 2i & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2+2i \\ -1+2i \end{pmatrix}$$

In bra ket notation this is

$$\widehat{A}|\psi\rangle = \frac{1}{\sqrt{2}}\Big[(2+2i)|\varphi_1\rangle - (1-2i)|\phi_2\rangle\Big].$$

Call this state vector $|\chi\rangle$.

The bra corresponding to $|\psi\rangle$ is represented by the row vector

$$\langle\psi| \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}$$

and hence

$$\begin{aligned}\langle\psi|\widehat{A} &\doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}2 & 2i\\ 2i & -1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}2+2i & -1+2i\end{pmatrix}\end{aligned}$$

In bra ket notation this is:

$$\langle\psi|\widehat{A} = \frac{1}{\sqrt{2}}\Big[(2+2i)\langle\varphi_1| - (1-2i)\langle\varphi_2|\Big].$$

(c) From above, we have $\widehat{A}|\psi\rangle = |\chi\rangle$ where

$$|\chi\rangle = \frac{1}{\sqrt{2}}\Big[(2+2i)|\varphi_1\rangle - (1-2i)|\phi_2\rangle\Big].$$

The bra vector corresponding to this is then

$$\langle\chi| = \frac{1}{\sqrt{2}}\Big[(2-2i)\langle\varphi_1| - (1+2i)\langle\varphi_2|\Big]$$

which is *not* the same as $\langle\psi|\widehat{A}$.
