PHYSICS 201 Interference and Diffraction/Wave Mechanics (2009)

Assignment 5

Due Date: Wednesday 20th May 2009

1. Four atoms are equidistantly placed in a line in an atom trap. The atoms are all simultaneously excited by a laser pulse and thereafter emit light of wavelength λ. Suppose the atoms are separated by a distance equal to this wavelength. Sketch a polar plot of the intensity of the light emitted by the atoms.

   [In practice, all the light comes out in a single flash (superradiance).]

2. In the Huygen's wavelength construction used to derive the single slit diffraction pattern, it is noted that the amplitude a of the sources of the Huygen's wavelets must be proportional to $1/N$ where N is the number of sources on the wavefront within the slit. It can also be shown that a must also be proportional to b, the width of the slit.

   (a) A single slit of width b is made into a double slit by obscuring its centre with an opaque strip of adjustable width c. Find expressions for the width and separation of the two slits so formed, and hence write down an expression for the Fraunhofer diffraction pattern formed if waves of wavelength λ are normally incident on the slits.

   (b) Show that it is not possible, by adjusting the width of the opaque strip, to cancel the first order interference fringe, but that the second order fringe can be cancelled.

3. Buckminsterfullerene molecules (buckyballs) are molecules made up of 60 carbon atom arranged to form a geodesic sphere. Suppose that a beam of buckyballs of width 5μm are sent at a velocity of 100 ms^{-1} through a diffraction grating created by a standing wave laser beam in which the 'slits' are separated by a distance of 150 nm. The buckyballs then strike an observation screen placed a further 1.25 m past the slits.

   (a) Calculate the de Broglie wavelength of the buckyballs (i.e. treat them as if they were quantum objects).

   (b) Assuming the molecules are point objects, estimate the distance between the maxima of the resultant interference pattern on the screen.

   (c) Estimate the number of slits in the diffraction grating created by the standing wave.

   (d) Given that a buckyball has a diameter of approximately 1 nm, how does the size of the buckyball compare with the width of each diffraction peak, and the distance between neighbouring maxima of the interference pattern. Is the size of the C_{60} molecule likely to effect the visibility of the interference fringes? [The width of each interference maximum is roughly the distance between the first minimum on either side of a principal maximum.]

   Reference: Olaf Nairz, Björn Brezger, Markus Arndt, and Anton Zeilinger, *Diffraction of Complex Molecules by Structures Made of Light*, Phys. Rev. Lett. **87**, 160401 (2001)

4. Electrons are fermions: they do not enjoy each other's company. This has important consequences for the evolution of stars as it gives rise to a very powerful pressure known as 'electron degeneracy pressure' which can help to prevent a star from collapsing under its own weight.

   (a) Assume that there are N electrons confined to a volume V. Show that the average distance between the electrons is

   $$d \approx \left(\frac{V}{N}\right)^{\frac{1}{3}}.$$

   (b) As electrons are fermions, they will tend to stay within a region of volume V/N.

      (i) Estimate the uncertainty Δx in the position of an electron in the gas.

      (ii) From the uncertainty principle $\Delta x \Delta p \geq \hbar/2$, estimate the momentum of each electron.

      (iii) Hence show that the electrons will have an estimated combined energy

      $$U = \frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{2/3} N.$$

      [Your result may vary from this depending on your estimates of uncertainty in position, and the exact form of the uncertainty relation used.]

   (c) Thermodynamics tells us that the pressure exerted by the electron gas will be given by

   $$P = -\frac{dU}{dV}.$$

   Use this to calculate the degeneracy pressure of the electron gas. Estimate this pressure in a stellar core where the number of electrons per unit volume can be on the order of 10^{30} m^{-3}.

5. The radius of the event horizon of a black hole is the radius at which the escape velocity of any particle equals the speed of light. The escape velocity of a particle a distance R from the centre of a spherical object of mass M (and radius less than R) can be shown to be, according to the Newtonian theory of gravitation

   $$v = \sqrt{\frac{2GM}{R}}.$$

   (a) Assuming this is valid even if the escape velocity is c, the speed of light, derive an expression for the radius of a black hole. (This entirely Newtonian argument gives the same answer as general relativity!)

   (b) Assume now that due to quantum mechanical processes, a black hole can emit black body radiation (Hawking radiation). The point of emission of a photon can be estimated to within an uncertainty $\Delta x \sim R$. Given that the energy of a photon is given by $E = pc$, estimate the energy of the emitted photon.

   (c) The theory of black body radiation says that the energy of the emitted photons are peaked near the value $k_B T$ where k_B is Boltzmann's constant and T is the absolute temperature of the black body. Show that this implies that the temperature of a black hole can be estimated to be

   $$T \sim \frac{\hbar c^3}{2k_B MG} \qquad \left[\text{exact result: } T = \frac{\hbar c^3}{4\pi k_B MG}\right]$$

(d) What would this temperature be for a black hole with a mass equal to that of

(i) the sun;

(ii) an electron?

($M_{\text{Sun}} = 2 \times 10^{30}$ kg, $G = 6.7 \times 10^{-11}$ $\text{Nm}^2\text{kg}^{-2}$, $m_e = 9 \times 10^{-31}$ kg.)