MACQUARIE UNIVERSITY

Department of Physics
Division of ICS

Mathematical Methods (2005)

Assignment 3 due 26 May

1. In the theory of the laser, there arises a recurrence relation for the probability of finding n photons inside a laser cavity at steady state operating and well above threshold, given by
$$-\gamma(nP_n - (n+1)P_{n+1}) - R(P_n - P_{n-1}) = 0$$
where $P_n = 0, n < 0$. Solve this recurrence relation by first solving the recurrence relation for the quantity $C_n = \gamma n P_n - RP_{n-1}$.

2. Find an explicit expression for the u_n satisfying
$$u_{n+1} + 5u_n + 6u_{n-1} = 2^n$$
given that $u_0 = u_1 = 1$. Deduce that $2^n - 26(-3)^n$ is divisible by 5 for all integer n.

3. For the differential equation
$$\frac{d^2y}{dz^2} + 2z\frac{dy}{dz} + 4y = 0$$
determine whether or not $z = 0$ is an ordinary point, a regular singular point, or an irregular singular point for the equation. By use of an appropriate power/Frobenius series substitution, show that one solution to the differential equation is
$$y_1(z) = ze^{-z^2}$$
and obtain a series expansion for the remaining solution.

4. (a) Determine the singular points of the differential equation
$$(1-z^2)\frac{d^2y}{dz^2} - 3z\frac{dy}{dz} + \lambda y = 0.$$
and determine whether they are regular or irregular singular points.
   (b) Find two power series about $z = 0$ of this differential equation. Deduce that the value of λ for which the corresponding power series becomes an Nth-degree polynomial $U_N(z)$ is $N(N+2)$. Construct $U_2(z)$ and $U_3(z)$.

5. Verify that $z = 0$ is a regular singular point of the equation
$$z^2y'' - \tfrac{3}{2}zy' + (1+z)y = 0$$
and that the indicial equation has roots 2 and $\frac{1}{2}$. Show that the general solution is
$$y(z) = 6a_0z^2\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)2^{2n}z^n}{(2n+3)!} + b_0\left(z^{\frac{1}{2}} + 2z^{\frac{3}{2}} - \tfrac{1}{4}z^{\frac{1}{2}}\sum_{n=2}^{\infty}\frac{(-1)^n2^{2n}z^n}{n(n-1)(2n-3)!}\right).$$

Updated: 19th May 2005 at 8:58am.