MACQUARIE UNIVERSITY

Department of Physics
Division of ICS

Mathematical Methods (2005)

Assignment 2 due 5 May

1. Consider the simple harmonic oscillator equation

$$\frac{d^2y}{dt^2} + y = 0$$

   (a) Write this equation as a matrix equation of the form

$$\frac{d\mathbf{V}}{dt} = \mathsf{M}\mathbf{V}$$

   where

$$\mathsf{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   (b) Show that $\mathsf{M}^2 = -\mathsf{I}$ where I is the identity matrix and hence show that

$$e^{\mathsf{M}t} = \mathsf{I} + \mathsf{M}t + \frac{\mathsf{M}^2}{2!}t^2 + \frac{\mathsf{M}^3}{3!}t^3 + \ldots$$

   can be written as

$$e^{\mathsf{M}t} = \mathsf{I}\cos t + \mathsf{M}\sin t.$$

   (c) Write down the solution for $\mathbf{V}$ in terms of the initial values of $y(0)$ and $\dot{y}(0)$ and hence write down the solution for $y(t)$.

2. The Abraham-Lorentz equation for a classical charged particle feeling the effects of its own electric field is given by

$$m\left(\frac{d^2x}{dt^2} - \tau\frac{d^3x}{dt^3}\right) = F$$

   where m is the mass of the particle, F is any externally applied force and

$$\tau = \frac{e^2}{6\pi\epsilon_0 mc^3}$$

   is a constant with the units of time with the value, for an electron, of 6.26×10^{-24} sec.[1]

   (a) Find the complementary function for this equation.

[1] The time τ is effectively the time for light to cross the 'width' of the particle. The width, or diameter of the particle, is roughly the size it must have so that the energy stored in the electric field of the particle is equal to its relativistic mass mc^2, i.e. the entire mass of the particle is due to the energy contained its its electric field.

(b) If the driving term is $F = qE_0 \sin \omega t$ determine the full solution of this equation expressed in terms of the initial values $x(0), \dot{x}(0)$, and $\ddot{x}(0)$.

(c) What is unphysical about this solution? For what initial conditions would this unphysical behaviour not occur?

3. (a) Write down the complementary function for the differential equation

$$\frac{d^2y}{dt^2} + y = \cos rt$$

and by substituting a suitable trial function, determine a particular integral for $r \neq 1$.

(b) Express the complete solution to this equation in terms of the initial values $y(0)$ and $\dot{y}(0)$.

(c) By taking the limit of $r \to 1$ of the *complete* solution, determine the solution of this equation for $r = 1$.

(d) Show that the same result follows if a trial solution $y_p = t(A \cos t + B \sin t)$ is used to determine the particular integral.

4. Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = (1+x)e^{-x}; \qquad y(0) = 0, \quad y'(0) = 1$$

by solving for the roots of the auxiliary equation to obtain the complementary function, and looking for an appropriate trial function to obtain a particular integral.

5. Consider the pair of coupled differential equations

$$\dot{x} = -2x + y$$
$$\dot{y} = -x + \cos t$$

By Laplace transform methods, find the solution of these equations given the initial $x(0) = y(0) = 0$.

6. (a) Show that the Wronskian $W(x) = y_1 y_2' - y_1' y_2$ of two independent solutions $y_1(x)$ and $y_2(x)$ of the differential equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

satisfies the equation

$$\frac{dW}{dx} = -P(x)W$$

where $P(x) = a_1(x)/a_2(x)$.

(b) Show that the Wronskian can also be written in the form

$$W(x) = y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right)$$

(c) Hence show that if one solution $y_1(x)$ is known, then the other solution $y_2(x)$ can be derived from the first via

$$y_2(x) = y_1(x) \int \frac{\exp\left[-\int P(x)dx\right]}{[y_1(x)]^2} dx.$$

(d) Given that $y(x) = 1/x$ is a solution of

$$F(x, y, y', y'') = x(x+1)y'' + (2 - x^2)y' - (2+x)y = 0$$

find a second linearly independent solution.