MACQUARIE UNIVERSITY

Department of Physics
Division of ICS

Mathematical Methods (2005)

Assignment 1 due 8 April

1. (a) Solve the first order differential equation

$$(1+x^2)\frac{dy}{dx} = 1 + y^2$$

with the initial condition $y = -1$ when $x = 0$.

(b) Determine an integrating factor that makes the equation

$$3x^2y + \alpha\left(x^3 + 2y^2\right)\frac{dy}{dx} = 0$$

exact and hence find the general solution to this equation for arbitrary α. Also determine the value of α for which this equation is exact.

2. Excited atoms undergoing spontaneous decay are replenished by a pumping mechanism at a rate that depends on the square of the number of atoms currently excited, such that the number of excited atoms satisfies

$$\frac{dN}{dt} = -\lambda N + \alpha N^2.$$

If there are initially N_0 excited atoms present, find $N(t)$ at later times, and comment on the results if λ/α is greater than, less than, or equal to N_0.

3. An object of mass m initially at rest at $z = z_0 > 0$, falls under gravity in a resistive medium. The motion satisfies the equation

$$m\frac{dv}{dt} = \eta(z)v^2 - mg$$

where $\eta(z)$ is the coefficient of resistance at a distance z into the medium. [Upwards is taken as the positive z direction.]

(a) Rewrite this equation such that the independent variable is z and the dependent variable is v.

(b) Show that the resultant equation is inexact, and determine the integrating factor.

(c) What is the velocity of the object as a function of *distance* if

$$\eta(z) = \tfrac{1}{2}\lambda\big(1 - \tanh[(\lambda/m)z]\big).$$

(d) What is the terminal velocity of the particle for η as given in part (c)?

4. According to Fermat's principle of optics, a light ray passing through a medium with a spatially dependent refractive index $n(x, y)$ will follow a path $y(x)$ for which

$$I[y(x)] = \int_{x_1,y_1}^{x_2,y_2} n(x, y)ds,$$

where $ds = \sqrt{(dx)^2 + (dy)^2}$, is a minimum.

A ray of light is travelling in the (x, y) plane through a medium of variable refractive index for which $n(y) = 2 - |y|$ for $-1 < y < 1$. The ray passes through the point $(0, 0)$ at an angle θ to the X-axis. The equation for the path is

$$\frac{2-y}{\sqrt{1 + (dy/dx)^2}} = A$$

where A is a constant.

One path passing through the point $(0, 0)$ will just graze the edge of the medium at $y = 1$ and will subsequently be refracted back into the medium. Find the equation for this path, and the angle θ.

5. A reflecting mirror is made in the shape of a surface of revolution generated by revolving the curve $y(x)$ about the $x-$axis. In order that light rays emitted from a point source at the origin are reflected back parallel to the $x-$axis, the curve $y(x)$ must obey

$$\frac{y}{x} = \frac{2p}{1 - p^2}$$

where $p = dy/dx$. By solving this equation for x, differentiating with respect to y, and solving the resultant differential equation, find the curve $y(x)$.

6. The motion of a particle is found to satisfy the equation

$$x = vt + \alpha v^2$$

where v is its velocity at time t. Given that at $t = 0$ the particle is stationary, determine its subsequent motion as a function of time. [Hint: Differentiate with respect to t.]